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ON THE EXISTENCE AND REGULARITY OF FUNDAMENTAL DOMAINS WITH LEAST BOUNDARY AREA

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Introduction

Let M be a three-dimensional compact smooth Riemannian manifold. Let Φ_0 be the set of all fundamental domains of M with Lipschitz boundary in $\tilde{M}$, the universal covering space of M. Then it is a question of basic interest to see whether one can find a fundamental domain in Φ_0 with least boundary area among all fundamental domains in Φ_0. Moreover, passing to subfamilies of Φ_0, one can ask similar questions: Let Φ_1 be the subfamily of Φ_0 consisting of all fundamental domains of M which are homeomorphic to an open ball, and let Φ_2 be the subfamily of Φ_1 consisting of all fundamental domains of M whose closures are homeomorphic to a closed ball. Can one find a fundamental domain in Φ_1, or Φ_2, whose boundary area (counting multiplicity) is the minimum among all fundamental domains in Φ_1, or Φ_2? These problems were proposed by Michael H. Freedman.

In this paper we answer the first problem, the case of Φ_0, in the affirmative (Theorem 3). We then discuss the second problem, the case of Φ_1, and derive an affirmative answer under the assumption that M is irreducible, that is, every embedded sphere in M bounds a ball (Theorem 5). The third problem, the case of Φ_2, remains open. Besides the existence of minimizing fundamental domains in Φ_0 and Φ_1, we also obtain the regularity of the boundaries of these minimizing fundamental domains (Theorem 4). If we define a spine to be a subset of M whose complement in M is homeomorphic to an open ball, then the second problem is equivalent to finding an area minimizing spine of M.

For a two-dimensional compact Riemannian manifold M^2 the problem is much simpler to solve and easier to visualize. In fact, any fundamental domain of M^2 with least boundary length among all fundamental domains is always homeomorphic to an open disk. Furthermore the boundary of a minimizing fundamental domain consists of geodesic segments of $\tilde{M}^2$ meeting each other at 120° angles, and the number of edges and vertices are both $6 - 6\chi(M)$

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(Proposition). The simplest example is the flat torus T^2. A minimizing fundamental domain of T^2 is not the square of Figure 1 but the hexagon of Figure 2 (see Appendix, 1). This is because a triple point is area minimizing under Lipschitz map without counting multiplicity whereas a quadruple point is not.

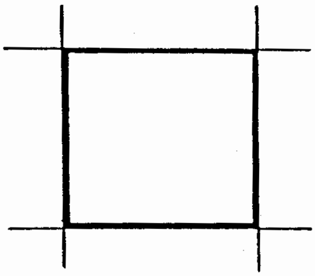


FIGURE 1

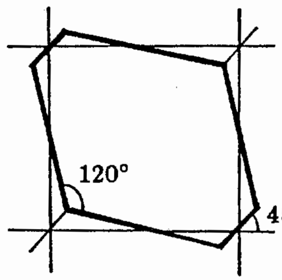


FIGURE 2

The methods used in this paper are those of geometric measure theory. In showing the existence of a minimizing fundamental domain for the first problem, we consider the characteristic functions of fundamental domains and use the compactness of functions of bounded variation. The existence for the second problem follows from the compactness of varifolds, viewing the boundaries of fundamental domains as 2-varifolds.

Throughout this paper we apply the cutting and pasting process extensively. In this process, however, in order not to change the topology of fundamental domains, we must assume that M is irreducible for the second problem. An example, the standard $S^2 \times S^1$, indicates that irreducibility is necessary: A fundamental domain $S^2 \times (0,1)$ in the universal covering space $S^2 \times \mathbf{R}^1$ of $S^2 \times S^1$ has least boundary area among the elements of Φ_0 (see Appendix, 2). But $\partial(S^2 \times (0,1))$ is also the varifold limit of $\{\partial F_k\}$, where the F_k's are fundamental domains in Φ_1 obtained by cutting out a slanted rod with thickness ε_k, $\varepsilon_k \to 0$ as $k \to \infty$, which connects $S^2 \times \{0\}$ to $S^2 \times \{1\}$, translating and pasting the rod to $S^2 \times (0,1)$ along $S^2 \times \{1\}$ (Figure 3). $\{F_k\}$ is also a minimizing sequence in Φ_2.

The main difficulty lies in controlling unbounded fundamental domains. Indeed if $\tilde{M}$ is noncompact, then the fundamental domains of M may be unbounded. Moreover, since an unbounded thin spike may have arbitrarily small boundary area, we can have a minimizing sequence of fundamental domains which are unbounded in $\tilde{M}$. This bad minimizing sequence is to be replaced by a uniformly bounded one by applying a cutting and pasting process (Theorem 2). To do so, we should verify that both Φ_0 and Φ_1 are closed

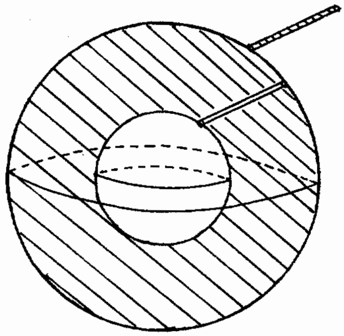


FIGURE 3

under cutting and pasting process, and that the boundary area of fundamental domains does not increase substantially after cutting and pasting. These are verified through an appropriate adaptation of [4] to our setting. This adaptation requires a great deal of care.

Once we have uniform boundedness of the minimizing sequence we can get the desired regularity results for the first problem as in [13]. For the second problem, we need to modify carefully the arguments in [13] to preserve the topology of fundamental domains. Thus we show that the projection into M of the boundary of a minimizing fundamental domain consists of minimal surfaces meeting each other at equal (120°) angles along Hölder continuously differentiable curves, like compound soap films, and four such curves meet each other at isolated points at which six sheets of minimal surfaces meet at equal angles. It should be mentioned that from [8] we can obtain analyticity of the singular curve in case $\tilde{M}$ is isometric to a Euclidean 3-space $\mathbf{R}^3$, and higher regularity in general.

Now we mention two outstanding problems: (i) What is a fundamental domain of a flat cubic torus $\mathbf{T}^3$ with least boundary area? (See Appendix 3.) (ii) If the curvature of M is nonpositive, does the minimizing fundamental domain of M in Φ_0 or Φ_1 also belong to Φ_2? Is the minimizer star-shaped?

Finally, we wish to express our sincere gratitude to Richard M. Schoen, who introduced us to geometry and analysis. Also we would like to thank Michael H. Freedman and Leon Simon for their interest and helpful discussions.

1. Terminology

In general, we will use the definitions and notation of [7] and [13] throughout.

(1) Let M be a three-dimensional compact smooth Riemannian manifold, $\tilde{M}$ the universal covering space of M, and p the projection map from $\tilde{M}$ onto M.

A *fundamental domain* of M is an open set F in $\tilde{M}$ satisfying

$$\mathrm{Vol}(F) = \mathrm{Vol}(p(F)) = \mathrm{Vol}(M).$$

Let Φ_0 denote the set of all fundamental domains of M with Lipschitz boundary and let Φ_1 denote the set of all fundamental domains in Φ_0 which are homeomorphic to an open ball. Then, for any $F \in \Phi_1$, $M \sim p(\partial F)$ is homeomorphic to an open ball. A subset S of M will be called a *spine* of M if $M \sim S$ is homeomorphic to an open ball. Note that, for any $F \in \Phi_1$, $p(\partial F)$ is a spine of M and conversely, for any Lipschitz spine we can find $F \in \Phi_1$ with $p(\partial F) = S$.

A fundamental domain $F \in \Phi_1$ will be said to be *reducible* if there exists a proper subset S of $p(\partial F)$ such that S has Lipschitz boundary and $M \sim S$ is still homeomorphic to an open ball. Let Φ denote the set of all fundamental domains in Φ_1 which are not reducible.

A fundamental domain $F \in \Phi$ will be said to be *adequate* if $\overline{F}$ is homeomorphic to a closed ball, otherwise F will be said to be *inadequate*.

Consider the subset R_F of ∂F which consists of $q \in \partial F$ with the property that there is an $r_q > 0$ such that for any geodesic ball $G_r(p(q))$ with center $p(q)$ and radius $r < r_q$, $p(F) \cap G_r(p(q))$ consists of two components only. Each component of R_F will be called a *face* of F. Any subset of $\partial F \sim R_F$ which is homeomorphic to an open interval $(0,1)$ will be called a *multiple curve* of F. Any point of $\partial F \sim R_F$ at which at least three distinct multiple curves of F meet each other will be called a *multiple point* of F.

For $F \in \Phi_0$, δF denotes $p(\partial F)$. The image under p of face, multiple curve, or multiple point of F will be called face, multiple curve, or multiple point of δF respectively.

For any set $K \subset M$, define $\tilde{K} = \{x \in \tilde{M} \colon p(x) \in K\}$.

(2) $B^n(p,r)$ and $U^n(p,r)$ will denote respectively the closed and open geodesic balls with radius r and center p in M^n or $\mathbf{R}^n$. B, B° will denote $B^3(0,1)$, $U^3(0,1)$ in $\mathbf{R}^3$ respectively.

We define $D = \{x \in \mathbf{R}^2 \colon |x| < 1\}$.

(3) For each $r > 0$ we define $\mu_r \colon \mathbf{R}^n \to \mathbf{R}^n$, $\mu_r(x) = rx$, $x \in \mathbf{R}^n$, and for each $p \in \mathbf{R}^n$ we define $\tau_p \colon \mathbf{R}^n \to \mathbf{R}^n$, $\tau_p(x) = x - p$, $x \in \mathbf{R}^n$.

(4) We say that S is *area minimizing in an open set* $U \subset M^n$ *under a Lipschitz map* provided

$$H^m(S) \leq H^m(\phi(S))$$

whenever ϕ is a Lipschitz map on M^n such that ϕ maps U into U and leaves $M \sim U$ fixed, where H^m denotes the m-dimensional Hausdorff measure.

A varifold V in M^n is said to be *area minimizing under diffeomorphism* provided

$$\mathbf{M}(V) \leq \mathbf{M}(\phi_{\#}V)$$

whenever ϕ is a diffeomorphism of M^n, where M denotes the mass.

(5) Define $Y^1 \subset \mathbf{R}^2$ to be a union of three half-lines joining at the origin with 120° angles to each other. Define $Y \subset \mathbf{R}^3$ by $Y = (Y^1 \times \mathbf{R}^1) \cap B^3(0,1)$ (see Figure 4).

Define $T \subset \mathbf{R}^3$ as the intersection with $B^3(0,1)$ of an infinite cone from the origin through the 1-skeleton of a regular tetrahedron with its center of mass at the origin (see Figure 5).

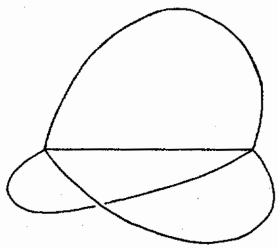


FIGURE 4

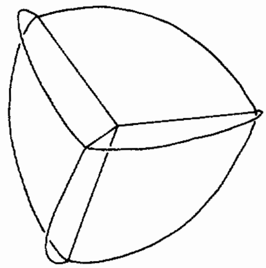


FIGURE 5

(6) For S a 2-rectifiable set, define $\sigma(S)$ as the singular set of S, that is, $\sigma(S)$ is the set of points in S at which there is no approximate tangent plane to S in $G(3,2)$, or $\Theta^2(H^2 \llcorner S, x)$ is not one. Define $\sigma_Y(S) \subset \sigma(S)$ as the points of density $3/2$, and $\sigma_T(S) \subset \sigma(S)$ as the points of density $\pi^{-1} 3 \cos^{-1}(-1/3)$. Define $R(S) = S \sim \sigma(S)$.

If V is a 2-varifold we define $\sigma_Y(V) = \sigma_Y(\operatorname{spt}\|V\|)$, $\sigma_T(V) = \sigma_T(\operatorname{spt}\|V\|)$, $R(V) = R(\operatorname{spt}\|V\|)$, and $\sigma(V) = \sigma(\operatorname{spt}\|V\|)$.

(7) Suppose S is a surface homeomorphic to D or ∂B. A *compound Jordan curve* in S is a connected union of a finite number of Jordan curves in S. Given a compound Jordan curve C in S, a Jordan curve $C_1 \subset C$ is said to be an *individual Jordan curve* of C if there exists a set $X \subset S \sim C$ with $X \approx D$ and $\partial X = C_1$. In case $S \approx D$, a subset C_2 of C is said to be an *outermost Jordan curve* of C if there exists an annular domain $Y \subset S \sim C$ with $C_2 = \partial Y \sim \partial S$.

(8) Recall that Φ_0 consists of fundamental domains of M with Lipschitz boundary. Hence, given $F \in \Phi_0$, there exists a family $\{S_i\}_{i \in I}$ of pairwise

disjoint subsets of δF $(= p(\partial F))$ with $\bigcup_{i\in I}\overline{S}_i = \delta F$ such that for all $i \in I$, S_i is C^1 up to its boundary and ∂S_i is a Lipschitz curve. Let X be a C^1 surface in M. Then we say that δF is *transversal* to X if S_i and ∂S_i are transversal to X for all $i \in I$. It follows from Sard's theorem that for almost all $r < i(M)$, the injectivity radius of M, δF intersects $\partial B^3(p,r)$ transversally and each component of $\delta F \cap \partial B^3(p,r)$ is a compound Jordan curve on $\partial B^3(p,r)$.

2. Replacement theorem and rectifiability

The following theorem guarantees that if U is an open convex subset of M, then we can always replace $F \in \Phi_0$ (respectively Φ) by $\hat{F} \in \Phi_0$ (respectively Φ) in such a way that $\delta\hat{F}$ satisfies an isoperimetric inequality in U.

Theorem 1 (*Replacement Theorem*). (1) *Given $U \approx B^\circ$ with C^1 boundary ∂U in M, and $F \in \Phi_0$ with δF intersecting ∂U transversally, there exists $\hat{F} \in \Phi_0$ such that*

(i) *$\delta\hat{F} \sim \overline{U} = \delta F \sim \overline{U}$;*

(ii) *$\delta\hat{F} \cap \overline{U} \subset \partial U$ and $H^2(\delta\hat{F} \cap \overline{U}) \leq c(H^1(\delta F \cap \partial U))^2$ for some $c > 0$ depending on U.*

(2) *Suppose M is irreducible. Given $U \approx B^0$ with C^1 boundary ∂U and $F \in \Phi$ with δF intersecting ∂U transversally, there exists $\hat{F} \in \Phi$ such that*

(i) *$\delta\hat{F} \sim U \subset \delta F \sim U$;*

(ii) *$\delta\hat{F} \cap U$ is a disjoint union of surfaces homeomorphic to D; if U is convex, then*

(ii)′ *$\delta\hat{F} \cap U$ is a disjoint union of area minimizing surfaces homeomorphic to D;*

(iii) *$H^2(\delta\hat{F} \cap U) \leq c(H^1(\delta\hat{F} \cap \partial U))^2$ for some $c > 0$ depending on U.*

For the proof of Theorem 1 we need the following two lemmas.

Lemma 1 (*Isoperimetric Inequality*). (1) *Let C_0 be a union of compound Jordan curves on a C^1 surface $S \approx \partial B$ in M and let $Y_0, \cdots, Y_l$ be open components of $S \sim C_0$. Suppose $\max_{0\leq i\leq l}\{H^2(Y_i)\} = H^2(Y_0)$. Then there exists $c > 0$ depending on S such that $H^2(S \sim Y_0) \leq c(H^1(C_0))^2$.*

(2) *Let C be a compound Jordan curve on a C^1 surface $S \approx \partial B$ in M and let $X_0, \cdots, X_n$ be the open components of $S \sim C$ where every X_i is homeomorphic to D. Suppose $\max_{0\leq i\leq n}\{H^2(X_i)\} = H^2(X_0)$. Then there exists $c > 0$ depending on S such that $H^2(S \sim X_0) \leq c(H^1(C))^2$.*

Proof of Lemma 1. Croke [5] showed that there exists a constant c depending on S such that whenever E is a region on S with $E \approx D$, then

$$\min\{H^2(E), H^2(S \sim E)\} \leq c(H^1(\partial E))^2.$$

We prove (2) first. Let $\beta = H^2(S)$ and $\alpha = H^2(X_0)$. First suppose $\alpha \le \beta/2$. Then

$$H^2(S \sim X_0) = \beta - \alpha \le \frac{\beta^2}{4\alpha} = c\left(\frac{1}{2}\sqrt{\frac{\alpha}{c}\frac{\beta}{\alpha}}\right)^2$$
$$= c\left(\frac{1}{2}\sum_{i=0}^{n}\frac{1}{\alpha}\sqrt{\frac{\alpha}{c}}H^2(X_i)\right)^2$$
$$\le c\left(\frac{1}{2}\sum_{i=0}^{n}\frac{1}{H^2(X_i)}\sqrt{\frac{H^2(X_i)}{c}}H^2(X_i)\right)^2$$
$$\le c\left(\frac{1}{2}\sum_{i=0}^{n}H^1(\partial X_i)\right)^2 = c(H^1(C))^2.$$

In case $\alpha > \beta/2$,

$$H^2(S \sim X_0) \le c(H^1(\partial X_0))^2 \le c(H^1(C))^2.$$

(1) Suppose $Y \subset S$ is a multiply connected open set with $H^2(Y) \le \beta/2$. First, assume there exists a component Z of $S \sim Y$ such that $Z \approx \overline{D}$ and $H^2(Z) > \beta/2$. Then

$$H^2(Y) \le H^2(S \sim Z) \le c(H^1(\partial Z))^2 \le c(H^1(\partial Y))^2.$$

Second, assume $H^2(Z_i) \le \beta/2$ for each component Z_i of $S \sim Y$. Then

$$H^2(Y) \le \sum_i H^2(Z_i) \le \sum_i c(H^1(\partial Z_i))^2 \le c\Big(\sum_i H^1(\partial Z_i)\Big)^2 = c(H^1(\partial Y))^2.$$

Therefore, in either case, $H^2(Y) \le c(H^1(\partial Y))^2$.

Let $\alpha_0 = H^2(Y_0)$ and suppose $\alpha_0 \le \beta/2$. Then, as in (2), we have

$$H^2(S \sim Y_0) \le c\left(\frac{1}{2}\sum_{i=0}^{l}\frac{1}{H^2(Y_i)}\sqrt{\frac{H^2(Y_i)}{c}}H^2(Y_i)\right)^2.$$

Since $H^2(Y_i) \le c(H^1(\partial Y_i))^2$ for all $i = 1, \cdots, l$, we have

$$H^2(S \sim Y_0) \le c\left(\frac{1}{2}\sum_{i=0}^{l}H^1(\partial Y_i)\right)^2 = c(H^1(C_0))^2.$$

If $\alpha_0 > \beta/2$, then

$$H^2(S \sim Y_0) = \sum_j H^2(Z_j),$$

where $\{Z_j\}$ is the set of all components of $S \sim Y_0$. Since $Z_j \approx \overline{D}$ and $H^2(Z_j) \le \beta/2$, we get

$$H^2(S \sim Y_0) \le \sum_j c(H^1(\partial Z_j))^2 \le c\Big(\sum_j H^1(\partial Z_j)\Big)^2 \le c(H^1(C_0))^2.$$

Lemma 2. (1) *Let $F \in \Phi_0$, and assume $U \approx B^\circ$ has C^1 boundary ∂U which is transversal to δF. Then there exists $\hat{F} \in \Phi_0$ with $\delta\hat{F} \sim \overline{U} = \partial F \sim \overline{U}$, $\delta\hat{F} \cap \overline{U} \subset \partial U$ such that $\partial U \sim (\delta\hat{F} \cap \overline{U})$ is a component of $\partial U \sim \delta F$.*

(2) *Assume M is irreducible, $F \in \Phi$, and E is a piecewise C^1 surface in M with $E \approx D$. If $\delta F \cap E$ is a compound Jordan curve disjoint from ∂E, then there exist $\hat{F} \in \Phi$ and a set $W \approx B^\circ$ in M such that $E \subset \partial W$, $\delta F \cap \partial W = \delta F \cap E$, and $\delta\hat{F} \subset (\delta F \sim W) \cup E$.*

Proof of Lemma 2. (1) Let Y be a set in F such that $p(Y)$ is a component of $\partial U \sim \delta F$. Let $\hat{U} \subset \tilde{M}$ be a component of $\tilde{U}$ with $Y \subset \partial\hat{U}$. Define $\hat{F} = (F \sim \overline{\tilde{U}}) \cup \hat{U} \cup Y$. Then we see that $\hat{F}$ is a fundamental domain of M in Φ_0. Obviously $\delta\hat{F} \sim \overline{U} = \delta F \sim \overline{U}$ and $\delta\hat{F} \cap \overline{U} = \partial U \sim p(Y)$.

(2) Since $\delta F \cap \partial E = \varnothing$ and $M \sim \delta F \approx B^\circ$, there exists a set $J \approx \overline{D}$ in M such that $\delta F \cap J = \varnothing$, $\partial J = \partial E$, and $J \cap \overline{E} = \partial E$. Then $J \cup E$ is homeomorphic to a sphere, and hence by the irreducibility hypothesis there exists a set $W \approx B^\circ$ in M with $\partial W = J \cup E$. Note that $\delta F \cap \partial W = \delta F \cap E$. Now let W_1 be the component of $\tilde{W}$ with $\partial W_1 \supset |F \cap \tilde{J}|$, and define $F_1 \in \Phi_0$ by $F_1 = (F \sim \tilde{W}) \cup W_1 \cup (F \cap \partial W_1)$. Then, since each component of $F \sim p^{-1}(\partial W)$ is homeomorphic to B°, so is each component of F_1. Let us assume that U_0 is the component of F_1 containing $F \cap \partial W_1$, and $U_1, \cdots, U_m$ are the remaining components of F_1. Then it is easy to see $\partial W \cap p(\overline{U}_i) \approx D$, $i = 1, \cdots, m$. Hence we can find $U_{i,0}$, a component of $p^{-1}(p(U_i))$, with $\partial W_1 \cap \overline{U}_{i,0} \approx D$ for each $i = 1, \cdots, m$. We thus paste each $U_{i,0}$ to U_0 along $\partial W_1 \cap \overline{U}_{i,0}$ to get a fundamental domain $\hat{F} \in \Phi$ with $\delta\hat{F} \subset (\delta F \sim W) \cup E$, and hence the proof is complete.

Proof of Theorem 1. (1) Let Y be a set in F such that $p(Y)$ is a component of $\partial U \sim \delta F$ with $H^2(p(Y)) = \max\{H^2(Y_i)\}$, where the maximum is taken over all components Y_i of $\partial U \sim \delta F$. Then Lemma 1(1) and Lemma 2(1) prove (1).

(2) Let K_i, $i = 1, \cdots, l$, be the components of $\delta F \cap \overline{U}$ and let C_{ij}, $j = 1, \cdots, m_i$, be the disjoint compound Jordan curves such that $K_i \cap \partial U = \bigcup_{j=1}^{m_i} C_{ij}$ for each i. Finally let $X_{ijk} \subset \partial U$, $1 \le k \le n_{ij}$, be such that $X_{ijk} \approx D$, $\partial U \sim C_{ij} = \bigcup_{k=1}^{n_{ij}} X_{ijk}$, for each i and j. Renumbering if necessary, one can assume $\max_{1 \le k \le n_{ij}}\{H^2(X_{ijk})\} = H^2(X_{ij1})$ for each i and j. Define $Y_{ij} = \partial U \sim \overline{X}_{ij1}$. Then $Y_{ij} \approx D$. Suppose $Y_{ab} \cap Y_{cd} \ne \varnothing$ for some $1 \le a, c \le l$,

$1 \leq b \leq m_a$, and $1 \leq d \leq m_c$. Since $C_{ab} \cap C_{cd} = \emptyset$, we have three possibilities:

$$Y_{ab} \subset Y_{cd}, \quad Y_{ab} \supset Y_{cd}, \quad Y_{ab} \cup Y_{cd} = \partial U.$$

The third case is not possible since otherwise we would get two contradictory conclusions: $H^2(X_{ab1}) < H^2(X_{cd1})$ and $H^2(X_{ab1}) > H^2(X_{cd1})$. The first two cases give us a partial ordering in the family $\{C_{ij}\}_{1 \leq i \leq l, 1 \leq j \leq m_i}$ in a standard way, i.e., $C_{ab} < C_{cd}$ if and only if $Y_{ab} \subset Y_{cd}$. Hence by choosing a minimal element successively from $\{C_{ij}\}$ one can get an ordered family $\{C_\alpha\}_{0 \leq \alpha \leq n}$ such that $\{C_\alpha\}_{0 \leq \alpha \leq n} = \{C_{ij}\}_{1 \leq i \leq l, 1 \leq j \leq m_i}$ and $C_\alpha < C_\beta$ only if $\alpha < \beta$. Define $Y_\alpha = Y_{ij}$ if $C_\alpha = C_{ij}$.

Let $\theta = \min_{0 \leq \alpha \neq \beta \leq n}\{\operatorname{dist}(C_\alpha, C_\beta)\}$. Then we can find $Z_0 \subset \partial U$ such that $Z \approx D$, $Z_0 \supset \overline{Y}_0$, and $HD(Z_0, Y_0) < \theta$. We thus apply Lemma 2 with Z_0 in place of E, to give $\hat{F} \in \Phi$ and a set $W_0 \approx B^\circ$ such that $Z_0 \subset \partial W_0$, $\delta F \cap \partial W_0 = \delta F \cap Z_0$, and $\delta \hat{F} \subset (\delta F \sim W_0) \cup Z_0$. W_0 plays the role of eliminating the piece of δF which lies either inside U or outside U. Hence we consider the following two cases:

Case 1. W_0 eliminates the piece of δF which lies outside U, i.e., there is an open set V_0 containing Y_0 such that $(\delta F \sim \delta \hat{F}) \cap V_0 \subset M \sim U$. Taking points of $\delta \hat{F} \cap \overline{Y}_0$ into U we obtain δF_1, $F_1 \in \Phi$, with $\delta F_1 \cap \partial U = (\delta \hat{F} \cap \partial U) \sim \overline{Y}_0$ ($\subset (\delta F \cap \partial U) \sim C_0$). Define $U_1 = U$.

Case 2. W_0 eliminates the piece of δF which lies inside U, i.e., there is an open set V_0 containing Y_0 such that $(\delta F \sim \delta \hat{F}) \cap V_0 \subset U$. Holding the set $\delta \hat{F} \sim Y_0$ fixed, and taking points of $\delta \hat{F} \cap Y_0$ into U we obtain δF_1, $F_1 \in \Phi$, with $\delta F_1 \cap \partial U \subset \delta F \cap \partial U$. Note that if K_0 is the component of $\delta F_1 \cap \overline{U}$ containing C_0, then $K_0 \sim C_0 = \bigcup_{i=1}^{m_1} Z_{1,i}$, where $Z_{1,i} \approx D$ and $\bigcup_{i=1}^{m_1} \partial Z_{1,i} = C_0$. Thus we can find an open set $U_1 \subset U$ such that $U_1 \approx B^\circ$ and $\delta F_1 \cap (U \sim U_1) = \bigcup_{i=1}^{m_1} Z_{i,1}$. Hence $\delta F_1 \cap \partial U_1 = (\delta F_1 \cap \partial U) \sim C_0 \subset \delta F \cap \partial U \sim C_0$.

In either case, note that $\delta F_1 \cap \partial U_1$ has fewer components than $\delta F \cap \partial U$, more precisely, $\delta F_1 \cap \partial U_1 \subset (\delta F \cap \partial U) \sim C_0$. We also have $\delta F_1 \sim U \subset \delta F \sim U$.

Now that we "eliminated" C_0 we again apply the same argument (cutting and slight perturbation) to $\delta F_1 \cap U_1$ to eliminate the remaining C_α's. First $\delta F_1 \cap \partial U_1$ can be viewed as a subsequence of $\delta F \cap \partial U$. Let C_{α_1} be the first element of this subsequence. Clearly $C_{\alpha_1} \neq C_0$. Define $Y_{\alpha_1,0} \subset \partial U_1$ by

$$Y_{\alpha_1,0} = \begin{cases} (Y_{\alpha_1} \sim (\partial U \sim \partial U_1)) \cup (\partial U_1 \sim \partial U) & \text{if } Y_{\alpha_1} \supset Y_0, \\ Y_{\alpha_1} & \text{if } Y_{\alpha_1} \cap Y_0 = \emptyset. \end{cases}$$

Then $Y_{\alpha_1,0} \approx D$ and hence we can find $Z_1 \subset \partial U_1$ such that $Z_1 \approx D$, $Z_1 \supset \overline{Y}_{\alpha_1,0}$, and $HD(Z_1, Y_{\alpha_1,0}) < \theta$. Now we can apply Lemma 2, with Z_1 in place of E, to give $\hat{F}_1 \in \Phi$ and a set $W_1 \approx B^\circ$ such that $Z_1 \subset \partial W_1$, $\delta F_1 \cap \partial W_1 = \delta F_1 \cap Z_1$, and $\delta \hat{F}_1 \subset (\delta F_1 \sim W_1) \cup Z_1$. Here again we consider the following two cases:

Case 1. $(\delta F_1 \sim \delta \hat{F}_1) \cap V_1 \subset M \sim U_1$ for some open set V_1 containing $Y_{\alpha_1,0}$. Taking points of $\delta \hat{F}_1 \cap \overline{Y}_{\alpha_1,0}$ into U we can obtain δF_2, $F_2 \in \Phi$ with $\delta F_2 \cap \partial U_1 = (\delta \hat{F}_1 \cap \partial U_1) \sim \overline{Y}_{\alpha_1}$. Define $U_2 = U_1$.

Case 2. $(\delta F_1 \sim \delta \hat{F}_1) \cap V_1 \subset U_1$. In this case holding the set $\delta \hat{F}_1 \sim Y_{\alpha_1}$ fixed, and taking the points of $\delta \hat{F}_1 \cap Y_{\alpha_1}$ into U_1 we can obtain δF_2, $F_2 \in \Phi$, with $\delta F_2 \cap \partial U_1 \subset \delta F_1 \cap \partial U_1$. Now we can find an open set $U_2 \subset U_1$ such that $U_2 \approx B^\circ$, and the component of $\delta F_2 \cap \overline{U}_1$ containing C_{α_1} is contained in $\overline{U}_1 \sim U_2$. Note that $\delta F_2 \cap (U_1 \sim U_2) = \bigcup_{i=1}^{m_2} Z_{2,i}$, where $Z_{2,i} \approx D$ and $\bigcup_{i=1}^{m_2} \partial Z_{2,i} = C_{\alpha_1}$.

In either case, $\delta F_2 \cap \partial U_2$ has fewer components than $\delta F_1 \cap \partial U_1$, i.e., $\delta F_2 \cap \partial U_2 \subset (\delta F_1 \cap \partial U_1) \sim C_{\alpha_1}$. Note also that $\delta F_2 \sim U_1 \subset \delta F_1 \sim U_1$.

Continuing the above procedure we can obtain for each $j = 1, \cdots, n_0 - 1$,

$$C_{\alpha_j} \quad \text{with } 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{n_0-1},\ n_0 - 1 < n$$
$$\text{(i.e., } \{C_{\alpha_j}\}_{0 \le j \le n_0-1} \text{ is a subsequence of } \{C_\alpha\}_{0 \le \alpha \le n});$$

$$\text{(2.1)} \quad U_{j+1} \quad \text{with } U_{j+1} \approx B^\circ \text{ and } U \supset U_1 \supset U_2 \supset \cdots \supset U_{n_0};$$
$$Y_{\alpha_j,0} \subset \partial U_j \quad \text{with } Y_{\alpha_j,0} \approx D, \text{ and}$$
$$Y_{\alpha_j,0} = \begin{cases} (Y_{\alpha_j} \sim (\partial U \sim \partial U_j)) \cup (\partial U_j \sim \partial U) & \text{if } Y_{\alpha_j} \supset Y_{\alpha_{j-1}}, \\ Y_{\alpha_j} \quad \text{if } Y_{\alpha_j} \cap Y_{\alpha_{j-1}} = \varnothing; \end{cases}$$

$$\text{(2.2)} \quad \begin{aligned} &Z_j \subset \partial U_j \quad \text{with } Z_j \approx D,\ Z_j \supset \overline{Y}_{\alpha_j,0}, \text{ and } HD(Z_j, Y_{\alpha_j,0}) < \theta; \\ &F_{j+1} \in \Phi \quad \text{with } \delta F_{j+1} \sim U_j \subset \delta F_j \sim U_j \ (U_0 = U, F_0 = F), \end{aligned}$$

such that we have the following additional properties:

$$C_{\alpha_j}, C_{\alpha_{j+1}}, \cdots, C_{\alpha_{n_0-1}} \subset \delta F_j \cap \partial U_j;$$
$$\delta F_{j+1} \cap \partial U_{j+1} \subset (\delta F_j \cap \partial U_j) \sim C_{\alpha_j};$$
$$\text{either } U_{j+1} = U_j \text{ or } U_{j+1} \text{ is a proper subset of } U_j, \text{ and}$$
$$\text{(2.3)} \quad \delta F_{j+1} \cap (U_j \sim U_{j+1}) = \bigcup_{i=1}^{m_{j+1}} Z_{j+1,i},$$
$$\text{where } Z_{j+1,i} \approx D \text{ and } \bigcup_{i=1}^{m_{j+1}} \partial Z_{j+1,i} = C_{\alpha_j}.$$

Note that $\partial F_{n_0} \cap U_{n_0} = \varnothing$, so that

$$\begin{aligned} \delta F_{n_0} \cap U &= [(\delta F_{n_0} \sim U_{n_0}) \cup (\delta F_{n_0} \cap U_{n_0})] \cap U = (\delta F_{n_0} \sim U_{n_0}) \cap U \\ &= [(\delta F_{n_0} \sim U_{n_0-1}) \cap U] \cup [\delta F_{n_0} \cap (U_{n_0-1} \sim U_{n_0})]. \end{aligned}$$

Using (2.2) repeatedly we thus have

$$\begin{aligned}
\delta F_{n_0} \cap U &\subset [(\delta F_{n_0-1} \sim U_{n_0-1}) \cap U] \cup [\delta F_{n_0} \cap (U_{n_0-1} \sim U_{n_0})] \\
&= [(\delta F_{n_0-1} \sim U_{n_0-2}) \cap U] \cup \bigcup_{j=0}^{1} [\delta F_{n_0-j} \cap (U_{n_0-j-1} \sim U_{n_0-j})] \\
&\subset [(\delta F_{n_0-2} \sim U_{n_0-2}) \cap U] \cup \bigcup_{j=0}^{1} [\delta F_{n_0-j} \cap (U_{n_0-j-1} \sim U_{n_0-j})] \\
&= [(\delta F_{n_0-2} \sim U_{n_0-3}) \cap U] \cup \bigcup_{j=0}^{2} [\delta F_{n_0-j} \cap (U_{n_0-j-1} \sim U_{n_0-j})] \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&= [(\delta F_1 \sim U) \cap U] \cup \bigcup_{j=0}^{n_0-1} [\delta F_{n_0-j} \cap (U_{n_0-j-1} \sim U_{n_0-j})] \\
&= \bigcup_{j=0}^{n_0-1} [\delta F_{j+1} \cap (U_j \sim U_{j+1})].
\end{aligned}$$

Hence (2.3) gives $\delta F_{n_0} \cap U \subset \bigcup_{j=1}^{n_1} \bigcup_{i=1}^{m_{\beta_j}} Z_{\beta_j,i}$, where $\{\beta_1, \cdots, \beta_{n_1}\} = \{j : U_j$ is a proper subset of $U_{j-1}\}$, $n_1 \leq n_0$, $Z_{k,i}$ is homeomorphic to D, and $\bigcup_{i=1}^{m_k} \partial Z_{k,i} = C_{\alpha_{k-1}}$. In fact, renumbering j if necessary, one can easily have

$$\delta F_{n_0} \cap U = \bigcup_{k=1}^{n_2} \bigcup_{i=1}^{m_k} Z_{k,i},$$

where $n_2 \leq n_1$ and $\{j : C_{\alpha_{j-1}} \subset \delta F_{n_0} \cap \partial U\} = \{1, 2, \cdots, n_2\}$. This proves (ii) if we let $F_{n_0} = \hat{F}$. (2.1) and (2.2) prove (i).

Now we assume U is convex. The next step in the argument involves replacing each $Z_{k,i}$ by an area minimizing surface $\hat{Z}_{k,i} \approx D$ with $\partial \hat{Z}_{k,i} = \partial Z_{k,i}$. Since the family $\{\hat{Z}_{k,i}\}$ is pairwise disjoint, we can get a fundamental domain $\hat{F} \in \Phi$ such that $\delta \hat{F} \sim U = \delta F_{n_0} \sim U$ and

$$\delta \hat{F} \cap U = \bigcup_{k=1}^{n_2} \bigcup_{i=1}^{m_k} \hat{Z}_{k,i}, \tag{2.4}$$

which proves (ii)$'$.

It remains to prove (iii). Let $X_{k,i} \subset \partial U$ be such that $X_{k,i} \approx D$, $\partial X_{k,i} = \partial \hat{Z}_{k,i}$, and $H^2(X_{k,i}) \leq H^2(\partial U \sim X_{k,i})$. Then from the way we defined Y_α from C_α (or Y_{ij} from C_{ij}) we deduce that $X_{k,1}, X_{k,2}, \cdots, X_{k,m_k}$ are pairwise disjoint and

$$H^2(X_{k,j}) \leq H^2 \left(\partial U \sim \bigcup_{i=1}^{m_k} X_{k,i} \right)$$

for each $j = 1, \cdots, m_k$. Hence by (2.4)

$$\begin{aligned} H^2(\delta\hat{F} \cap U) &= \sum_{k=1}^{n_2}\sum_{i=1}^{m_k} H^2(\hat{Z}_{k,i}) \leq \sum_{k=1}^{n_2}\sum_{i=1}^{m_k} H^2(X_{k,i}) \\ &\leq \sum_{k=1}^{n_2} c\left(H^1\left(\bigcup_{i=1}^{m_k} \partial X_{k,i}\right)\right)^2 = \sum_{k=1}^{n_2} c\left(H^1\left(\bigcup_{i=1}^{m_k} \partial Z_{k,i}\right)\right)^2 \\ &= \sum_{k=1}^{n_2} c(H^1(C_{\alpha_{k-1}}))^2 \leq c(H^1(\delta\hat{F} \cap \partial U))^2. \end{aligned}$$

This completes the proof of Theorem 1.

The following lemma is a generalization of the Filigree Lemma in [4]. Here the word "filigree" means (very roughly) a collection of threadlike protrusions from a surface. For example, if $F \in \Phi$ satisfies $H^2(\delta F \cap B^3(p,r)) = \varepsilon r^2$, where ε is small, then $\delta F \cap B^3(p, r/2)$ would be classified as filigree.

The following lemma will enable us to "cut off" such sets under appropriate circumstances.

Lemma 3 (*Filigree lemma*). *Let $U_t = B^3(p, rt)$, $p \in M$, $r > 0$, $0 \leq t \leq 1$, and suppose U_t is convex for all $0 \leq t \leq 1$. Suppose also that there is a constant $c < \infty$ such that, whenever $E \subset \partial U_t$ is a set homeomorphic to D, then* $\min\{H^2(E), H^2(\partial U_t \sim E)\} \leq c(H^1(\partial E))^2$.

Finally, suppose $F \in \Phi_0$ and $\varepsilon > 0$ are such that

$$H^2(\delta F) \leq H^2(\delta G) + \varepsilon \quad \textit{for any } G \in \Phi_0. \tag{2.5}$$

Then $H^2(\delta F \cap U_t) \leq \varepsilon$ whenever $t \leq 1 - (2/r)\sqrt{c}\sqrt{H^2(\delta F \cap U_1)}$.

Moreover we can obtain the same result for $F \in \Phi$ with the additional assumption that M is irreducible.

Proof. We will prove the lemma in the case of Φ only since the proof for Φ_0 is basically the same. By Sard's Theorem δF intersects ∂U_t transversally for almost all $t \in (0,1)$. Then for almost all $t \in (0,1)$ we can apply Theorem 1, with U_t in place of U, to give $\hat{F} \in \Phi$ such that

$$\delta\hat{F} \sim U_t \subset \delta F \sim U_t, \tag{2.6}$$

$$H^2(\delta\hat{F} \cap U_t) \leq c(H^1(\delta\hat{F} \cap \partial U_t))^2. \tag{2.7}$$

By (2.5) we have

$$H^2(\delta F \cap U_t) + H^2(\delta F \sim U_t) \leq H^2(\delta\hat{F} \cap U_t) + H^2(\delta\hat{F} \sim U_t) + \varepsilon,$$

which together with (2.6) implies

$$H^2(\delta F \cap U_t) \leq H^2(\delta\hat{F} \cap U_t) + \varepsilon.$$

Then (2.7) gives

$$H^2(\delta F \cap U_t) \le c(H^1(\delta \hat{F} \cap \partial U_t))^2 + \varepsilon.$$

Since (2.6) yields $\delta \hat{F} \cap \partial U_t \subset \delta F \cap \partial U_t$, we have

$$H^2(\delta F \cap U_t) \le c(H^1(\delta F \cap \partial U_t))^2 + \varepsilon. \tag{2.8}$$

We can now suppose $H^2(\delta F \cap U_1) > \varepsilon$, otherwise the required conclusion is trivial. Then let $t_0 = \inf\{t\colon H^2(\delta F \cap U_t) > \varepsilon\}$ and define $f(t) = H^2(\delta F \cap U_t) - \varepsilon$, $t \in [t_0, 1]$. By the co-area formula we see that (2.8) implies

$$f(t) \le \frac{c}{r^2}\left(f'(t)\right)^2 \qquad \text{a.e. } t \in [t_0, 1].$$

Integrating this inequality (using the fact that $f(t)$ is an increasing function of t), we obtain

$$\sqrt{f(t_0)} \le \sqrt{f(1)} - \frac{r(1-t_0)}{2\sqrt{c}}.$$

However $f(1) = H^2(\delta F \cap U_1) - \varepsilon < H^2(\delta F \cap U_1)$; hence we deduce

$$1 - t_0 < \frac{2}{r}\sqrt{c}\sqrt{H^2(\delta F \cap Y_1)}.$$

That is,

$$t_0 > 1 - \frac{2}{r}\sqrt{c}\sqrt{H^2(\delta F \cap Y_1)},$$

and the required result is proved.

With the above filigree lemma we are now able to get the first regularity result as follows:

First let $\{F_k\}$ be a minimizing sequence in Φ_0 (respectively Φ), that is, $H^2(\delta F_k) \le H^2(\delta G) + \varepsilon_k$ for any $G \in \Phi_0$ (respectively Φ), where $\varepsilon_k \to 0$ as $k \to \infty$. Then we can apply compactness of varifolds and hence assume, by taking a subsequence if necessary, that there is a 2-varifold Δ in M such that $\Delta = \lim_{k\to\infty} |\delta F_k|$. Δ is of course area minimizing in M under diffeomorphism because $\mathbf{M}(\Delta) \le \mathbf{M}(\phi_{\#}\Delta)$ whenever ϕ is a diffeomorphism of M. Secondly we want to show that there is a constant $c_0 > 0$ such that whenever $p \in \operatorname{spt}\|\Delta\|$

$$\Theta^2(\|\Delta\|, p) \ge c_0. \tag{2.9}$$

Suppose $p \in \operatorname{spt}\|\Delta\|$ and let c be a constant as defined in the filigree lemma (obviously such c exists and depends only on M and r). By the filigree lemma we know that if $H^2(\delta F_k \cap B^3(p,r)) \le r^2/(16c)$, then $H^2(\delta F_k \cap B^3(p, r/2)) < \varepsilon_k$. If there is a subsequence $\{k'\} \subset \{k\}$ with $H^2(\delta F_{k'} \cap B^3(p,r)) \le r^2/(16c)$, then we would have $\operatorname{spt}\|\Delta\| \cap B^3(p, r/2) = \varnothing$, thus contradicting the fact that $p \in \operatorname{spt}\|\Delta\|$. Hence for all sufficiently large k we have $H^2(\delta F_k \cap B^3(p,r)) \geq r^2/(16c)$, from which we deduce

$$\|\Delta\|(B^3(p,r)) \geq \frac{1}{16c} r^2. \tag{2.10}$$

Thus we obtain (2.9) with $c_0 = 1/(16c)$. In particular, Δ is rectifiable by [1,5.5].

Hence we have proved the following.

Corollary 1 (*Rectifiability*). *Suppose $\{F_k\}$ is a minimizing sequence in Φ_0. Then a subsequence of the corresponding varifolds $|F_k|$ converges to a rectifiable 2-varifold Δ in M which is area minimizing under diffeomorphism and has $M(\Delta) = \inf\{H^2(\delta G) \colon G \in \Phi_0\}$. Moreover we can obtain the same result for $F \in \Phi$ assuming that M is irreducible.*

Lemma 4 (*Monotonicity lemma*). *Let Δ be a rectifiable 2-varifold in M which is area minimizing under diffeomorphism of M, and $p \in \operatorname{spt}\|\Delta\|$. Let ρ be the injectivity radius of M. Then there exists a function $\xi(r) = cr^m$, $c, m > 0$, such that the function $g \colon (0, \rho/2) \to \mathbf{R}^1$ defined by $g(r) = r^{-2}\|\Delta\|(B^3(p,r))e^{\xi(r)}$ is monotonically nondecreasing.*

Proof. Let $p⋇(\operatorname{spt}\|\Delta\| \cap \partial B^3(p,r))$ be the set of all geodesics from p to the points of $\operatorname{spt}\|\Delta\| \cap \partial B^3(p,r)$. Then there exists a sequence $\{\psi_n\}$ of "shrinking" diffeomorphisms of M such that $\psi_n(x) = x$ for all n and $x \in M \sim B^3(p,r)$, and

$$\operatorname{spt}\|\lim_{n\to\infty}(\psi_{n\#}\Delta)\| \cap B^3(p,r) = p⋇(\operatorname{spt}\|\Delta\| \cap \partial B^3(p,r)).$$

Since Δ is rectifiable, there is a positive H^2-measurable function θ on $N \equiv \operatorname{spt}\|\Delta\|$ such that

$$\Delta(S) = \int_{S\cap N} \theta \, dH^2 \quad \text{for any } H^2\text{-measurable } S. \tag{2.11}$$

Thus, as in [11, §15], we adapt the notation $\Delta = \mathbf{v}(N, \theta)$, which is characterized by (2.11). Note that for almost all $r > 0$, $N \cap \partial B^3(p,r)$ is rectifiable. Hence a 1-varifold $\Delta \cap \partial B^3(p,r)$ defined by

$$\Delta \cap \partial B^3(p,r) = \mathbf{v}(N \cap \partial B^3(p,r), \theta|_{\partial B^3(p,r)})$$

is a rectifiable varifold for almost all r.

Define $p⋇(\Delta \cap \partial B^3(p,r)) = \mathbf{v}(p⋇(N \cap \partial B^3(p,r), \tilde{\theta}))$, where $\tilde{\theta}(\tilde{x}) = \theta(x)$ whenever $\tilde{x}$ lies on the geodesic from p to $x \in N \cap \partial B^3(p,r)$. Then we can deduce that for almost all r,

$$\lim_{n\to\infty}(\psi_{n\#}\Delta) \llcorner (B^3(p,r) \times G(3,2)) = p⋇(\Delta \cap \partial B^3(p,r)).$$

Define $m(r) = \|\Delta\|(B^3(p,r))$. Since Δ is area minimizing,

$$m(r) \leq \|\psi_{n\#}\Delta\|(B^3(p,r)).$$

Taking the limit as $n \to \infty$, we have $m(r) \le \mathbf{M}(p \times\!\!\!\times (\Delta \cap \partial B^3(p,r)))$.

Now we can find a function $\xi(r) = cr^m$, $c, m > 0$, depending on M such that

$$M(p \times\!\!\!\times (\Delta \cap \partial B^3(p,r))) \le (1+\xi(r))\frac{r}{2} \cdot \mathbf{M}(\Delta \cap \partial B^3(p,r)).$$

Hence $m(r) \le \frac{1}{2}(1+\xi(r))r \cdot \mathbf{M}(\Delta \cap \partial B^3(p,r))$. Since $m'(r)$ exists and $\mathbf{M}(\Delta \cap \partial B^3(p,r)) \le m'(r)$, for almost all r, we get

$$m(r) \le \frac{1}{2}(1+\xi(r))rm'(r), \quad \text{i.e.,} \ \frac{r}{2}m'(r) - m(r)(1-\xi(r)) \ge 0.$$

It follows that $rm(r)\frac{d}{dr}\log g(r) \ge 0$ and hence $g'(r) \ge 0$.

Corollary 2. *Let Δ be an area minimizing 2-varifold in M obtained as above and $p \in \operatorname{spt} \|\Delta\|$. Then tangent cones to Δ exist at p; the tangent cones are cones in $\mathbf{R}^3$ and are area minimizing under diffeomorphism in $\mathbf{R}^3$.*

Proof. By the Nash embedding theorem we can assume that M is isometrically embedded in $\mathbf{R}^n$ for some $n \ge 3$. For convenience of notation we will assume $p = 0$. Let $\{r_i\}$ be a sequence of positive radii with $\lim_{i\to\infty} r_i = 0$. By Lemma 4 the varifolds

$$\Delta_i = ((\mu_{1/r_i})_\# \Delta) \llcorner (B^n(0,r_i) \times G(n,2))$$

all have bounded masses; since their supports also all lie in a bounded region of $\mathbf{R}^n$, the varifolds $\{\Delta_i\}$ have a convergent subsequence and a limit varifold μ. By definition μ is a tangent cone. Now $\mu \llcorner (B^n(0,1) \times G(n,2))$ is stationary, since any diffeomorphism of $\mathbf{R}^n$ which would decrease the mass of μ would also decrease uniformly the masses of the varifolds Δ_i for large i, contradicting the area minimizing property of Δ. The density ratios of μ are uniformly bounded away from 0 at each point in $\operatorname{spt} \|\mu\|$ since they are uniformly so bounded for Δ_i by (2.9). Therefore by [1, 5.5] $\mu \llcorner (B^n(0,1) \times G(3,2))$ is a rectifiable varifold and the support of $\|\mu\|$ is a rectifiable set [1, 2.8]. μ has density at every point in its support at least $(16c)^{-1}$, since each of the varifolds Δ_i does by (2.10). Therefore we may apply [1, 5.2, 6.5] to conclude that μ is in fact a cone. Finally μ is area minimizing in $\mathbf{R}^n$ because any diffeomorphism of $\mathbf{R}^n$ which saved mass in μ would also save mass uniformly in the varifolds Δ_i for large i. μ is obviously area minimizing in $\mathbf{R}^3$ too.

3. Uniformly bounded minimizing sequence

In order to obtain the desired existence and regularity result for both first and second problems, it is desirable to replace the given minimizing sequence of fundamental domains in Φ_0 or Φ by another minimizing sequence (with the same limit) of uniformly bounded ones in Φ_0 or Φ without filigrees.

In this section we use a cutting and pasting argument extensively. In this process however we have to be very careful not to change the topology of fundamental domains in the case of the second problem. The following lemma allows us to cut and paste fundamental domains without changing their topology.

Lemma 5 (*Pasting lemma*). (1) *Let F be an adequate fundamental domain in Φ. If there exists a face X of F which is multiply connected, then M is not irreducible.*

(2) *Let M be irreducible. In case F is an inadequate fundamental domain in Φ, there may exist a face X of F which is multiply connected. But then X can be replaced by a face homeomorphic to D in the following sense: For any nonnull-homotopic Jordan curve C on X, there exist $\hat{F} \in \Phi$ and $U \subset M$ with $U \approx B^\circ$ such that $\partial U \cap \delta F = p(C)$, $\delta\hat{F} \sim U = \delta F \sim U$, and $\delta\hat{F} \cap U \subset p(E)$, where E is a face of $\hat{F}$ homeomorphic to D.*

Proof. (1) By assumption there exists a face $Y \neq X$ of F such that $X = g(Y)$ for some translation g. Let S_1 and S_2 be such that S_1, $S_2 \subset \overline{F}$, S_1, $S_2 \approx \overline{D}$, $\partial S_1 = S_1 \cap \partial F \subset X$, $\partial S_2 = S_2 \cap \partial F \subset Y$, $p(\partial S_1) = p(\partial S_2)$, ∂S_1 and ∂S_2 are not null-homotopic in X and Y respectively, and $S_1 \cap S_2 = \varnothing$. Then $p(S_1 \cap S_2)$ is a closed surface in M homeomorphic to a sphere. Suppose M is irreducible. Define $S = p(S_1 \cup S_2)$ and let K be the component of $M \sim S$ which is homeomorphic to B°. Then $S_1 \cup g(S_2)$ is also a closed surface in $\tilde{M}$ homeomorphic to a sphere. Let $\hat{K}$ be the component of $\tilde{K}$ with $\partial\hat{K} = S_1 \cup g(S_2)$. Now since by hypothesis $\hat{K} \sim (F \cup g(F))$ is nonempty, we can find a translation h such that $h(F) \subset \hat{K} \sim (F \cup g(F))$. We then have

$$H^3(M) = H^3(h(F)) < H^3(\hat{K}) < H^3(M),$$

which is a contradiction. Thus M is not irreducible.

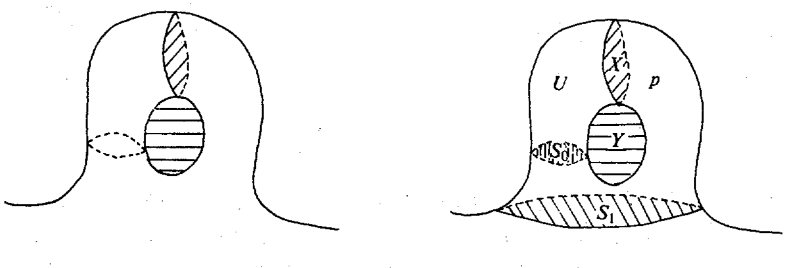
(2) As in the proof of (1) we can find Y, g, S_1, S_2, and K satisfying the same properties. Here we assume $\partial S_1 = C$. Since $S_1 \cup S_2$ divides F into three components, $\tilde{K} \cap F$ is either connected or $\tilde{K} \cap F$ consists of two components. Suppose $\tilde{K} \cap F$ is connected and let L_1, L_2 be the remaining components of $F \sim (S_1 \cup S_2)$, i.e., $F \sim (S_1 \cup S_2) = (\tilde{K} \cap F) \cup L_1 \cup L_2$. Clearly L_1, $L_2 \approx B^\circ$. Define $Z = \partial F \cap \tilde{K}$. Then we have $H^2(Z) > 0$ and $p(Z) \subset K$. We thus note that since K, $p(L_1)$, $p(L_2)$ are all homeomorphic to B°, $K \cup p(L_1) \cup p(L_2) \cup p(S_1^\circ) \cup p(S_2^\circ)$ is also homeomorphic to B°, where $S_i^\circ = S_i \sim \partial S_i$. Note also that

$$M \sim (K \cup p(L_1) \cup p(L_2) \cup p(S_1^\circ) \cup p(S_2^\circ)) = \delta F \sim p(Z),$$

since $p(Z) \subset K$. Hence $\delta F \sim p(Z)$ is a spine of M which is a proper subset of δF. Therefore F is reducible. Since this contradicts the hypothesis that F is in Φ, $\tilde{K} \cap F$ must consist of two components.

Define again $Z = \partial F \cap \tilde{K}$ and $L = F \sim (S_1 \cup S_2 \cup \tilde{K})$. Then both $K \cup p(L) \cup p(S_1^\circ)$ and $K \cup p(L) \cup p(S_2^\circ)$ are homeomorphic to B°. Let $\hat{F}_i \in \Phi$ be such that $p(\hat{F}_i) = K \cup p(L) \cup p(S_i^\circ)$, $i = 1, 2$. Then we see that $\delta \hat{F}_i = M \sim (K \cup p(L) \cup p(S_i^\circ))$, $\delta \hat{F}_i \sim \overline{K} = \delta F \sim \overline{K}$, $\delta \hat{F}_1 \cap \overline{K} = p(S_2)$, and $\delta \hat{F}_2 \cap \overline{K} = p(S_1)$. Now we can find $U \supset K$ with $U \approx B^\circ$ and $\partial U \cap \delta \hat{F}_i = p(C)$. Finally let $\hat{F}$ be either $\hat{F}_1$ or $\hat{F}_2$. Then U and $\hat{F}$ satisfy the desired properties.

Remark 1. (1) One can conclude from the above lemma that Φ is "closed" under cutting and pasting provided M is irreducible: Any $S \subset \overline{F}$ with $S \approx \overline{D}$ and $S \cap \partial F = \partial S$ cuts F into two components, V_1 and V_2. Suppose E_1, E_2 are faces of F with $E_i \subset \partial V_i$, $i = 1, 2$, and $E_1 = g(E_2)$ for some translation g. We then translate V_2 via g and paste $g(V_2)$ to V_1 along E_1. The resulting fundamental domain $\hat{F} \equiv V_1 \cup g(V_2) \cup E_1$ is homeomorphic to B° by (1) of the above lemma in case F is adequate, and by (2) in case F is inadequate. Hence $\hat{F} \in \Phi_1$. Since $\hat{F}$ may be reducible, we may have to eliminate the appropriate face (redundant face) of $\hat{F}$ to obtain a fundamental domain which is not reducible. Figure 6 illustrates two pathological cases; an inadequate fundamental domain with an annular face and a reducible fundamental domain with a redundant face (review terminology (1)).



(a) (b)

FIGURE 6 (in M)

Figures 6(a) and 6(b) are basically the same pictures of a part of δF in M which is topologically a punctured torus with two disks X and Y added. The two shaded disks, X and Y, are faces of δF. F is inadequate since $\overline{F}$ has two solid handles corresponding to the interior and the exterior of the (punctured) torus. $\partial X \cap \partial Y$ ($\equiv \{p\}$) is a multiple point of δF, and $\partial X \cup \partial Y \sim \{p\}$ is the union of two multiple curves of δF. The face of δF on which the dotted circle ($= \partial S_0$) lies is homeomorphic to an annulus. In Figure (b) S_0 cuts $M \sim \delta F$ into two components, U and $M \sim (\delta F \cup U)$. So we paste U to $M \sim (\delta F \cup U)$ along $\partial U \sim (X \cup S_0)$ to get a fundamental domain $\hat{F}$. But $\hat{F}$ is reducible since the face Y of $\delta\hat{F}$ is redundant (i.e., $\delta\hat{F} \sim Y$ is still a spine of M). The fundamental domain F' with $\delta F' = \delta\hat{F} \sim Y$ will then be not reducible, or $F' \in \Phi$. On the other hand, S_1 cuts $M \sim \delta F$ into $W \supset U$ and $M \sim (\delta F \cup W)$. Although W has an annular face $\partial W \sim (\overline{X} \cup \overline{Y} \cup \overline{S}_1)$, we can paste W to $M \sim (\delta F \cup W)$ along the annular face and eliminate (i.e., fill up) $\overline{X}$ and $\overline{Y}$ to obtain $\hat{F} \in \Phi$ with $\delta\hat{F} = [\delta F \sim (\partial W \cup Y)] \cup S_1$. This illustrates Lemma 5(2).

(2) Unlike Φ, Φ_2 is not closed under cutting and pasting: Let F be a fundamental domain of M in Φ_2. Suppose $E \subset F$, $E \approx D$, and $\partial E \subset \partial F$. Then E cuts F in two components U and V. Suppose X, Y, Z, W are faces of F with X, $Y \subset \partial V$ and Z, $W \subset \partial U$ such that $Z = \tau(X)$ and $W = \tau(Y)$ for some translation τ. Then $U \cup \tau(V) \cup Z$ is a fundamental domain in Φ but not in Φ_2 since the closure of $U \cup \tau(V) \cup Z$ is topologically a solid torus, as is illustrated by Figure 7.

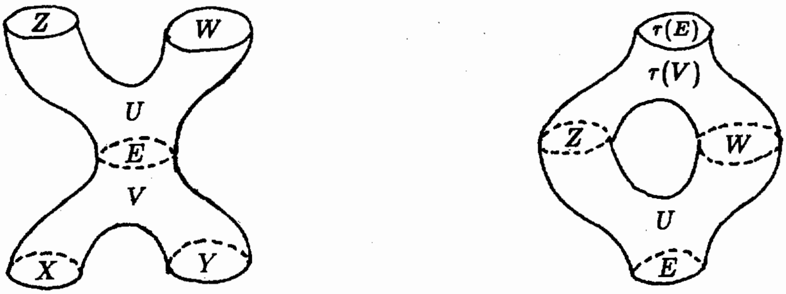


FIGURE 7 (in M)

Theorem 2 (*Boundedness theorem*). *Suppose $\{F_k\}$ is a sequence in Φ_0 such that $H^2(\delta F_k) \leq H^2(\delta G) + \varepsilon_k$ for any $G \in \Phi_0$, where $\varepsilon_k \to 0$ as $k \to \infty$. Then $\{F_k\}$ can be replaced by another minimizing sequence $\{F_k^4\}$ in Φ_0 which is uniformly bounded in $\tilde{M}$. Moreover we can obtain the same result for a minimizing sequence in Φ if M is irreducible.*

Proof of Theorem 2. We will prove this theorem in the case of Φ only, and then we will be able to see that the same argument is valid for the case of Φ_0 since the cutting and pasting method in Φ_0 (Lemma 2(1)) is much simpler than that in Φ (Lemma 5, Remark 1(1)).

In this proof we concern ourselves mainly with homothetic expansion and the cutting and pasting argument. Therefore we prove the theorem first assuming that M is locally isometrically in $\mathbf{R}^3$, and later we shall see that the theorem in general can be proved similarly by using the exponential map.

Suppose $p = 0$ for convenience. We write

$$K_{\rho,\sigma} = \{x \in B^3(0,\rho) \colon \operatorname{dist}(x, \operatorname{spt}\|\mu\|) \leq \sigma\}.$$

By definition of μ, we know there is a sequence $\{r_k\} \to \infty$ such that $\mu_{r_k\#}\Delta \to \mu$ as $k \to \infty$. By (2.10) it is then clear that for any $\sigma_0 \in (0,1)$ we can find r such that

$$B^3(0,1) \cap \operatorname{spt} \|\mu_{r\#}\Delta\| \subset K_{1,\sigma_0/2}. \tag{3.1}$$

Define $J_{\rho,\sigma} = U^3(0,\rho) \sim K_{\rho,\sigma}$ and $L_\delta = \{x\colon x \in J_{1,\sigma_0/2}, \operatorname{dist}(x, \partial J_{1,\sigma_0/2}) = \delta\}$ for $\delta > 0$. Then from (3.1) and the co-area formula it follows that for almost all $\sigma \in (\sigma_0/2, 1/2)$, $H^1(\mu_r(\delta F_k) \cap L_{\sigma-\sigma_0/2}) \to 0$ as $k \to \infty$. Thus for any given $\eta > 0$ we can assert that, for sufficiently large k, there is a $\sigma_k \in (3/4\sigma_0, \sigma_0)$ such that

$$H^1(\mu_r(\delta F_k) \cap L_{\sigma_k-\sigma_0/2}) < \eta. \tag{3.2}$$

We can also arrange, by Sard's theorem, that $\mu_r(\delta F_k)$ intersects $L_{\sigma_k-\sigma_0/2}$ transversally. Now we claim the following lemma.

Lemma 6. *For any $\sigma_0 \in (0,1)$ there is an $r > 0$ such that every filigree, or spike, of $\mu_r(\delta F_k)$ passing through $J_{1-\sigma_k+\sigma_0/2,\sigma_k}$ can be eliminated to make a new minimizing sequence $\{\delta F_k^2\}$ with the same varifold limit Δ. (That is, every filigree of $\mu_r(\delta F_k)$ passing through $J_{1-\sigma_k+\sigma_0/2,\sigma_k}$ can be cut off without changing varifold limit Δ.)*

Proof of Lemma 6. Choose r as in (3.1). Note that

$$L_{\sigma_k-\sigma_0/2} = \partial(J_{1-\sigma_k+\sigma_0/2}, \sigma_k).$$

Hence we apply Theorem 1 repeatedly with $\mu_r(\delta F_k)$ in place of δF, $\mu_r(F_k)$ in place of F, and each component of $J_{1-\sigma_k+\sigma_0/2,\sigma_k}$ in place of U. Then we get $F_k^1 \in \Phi$ and $Z_1, Z_2, \cdots, Z_{n_k}$ which are homeomorphic to D such that

$$\mu_r(\delta F_k^1) \cap J_{1-\sigma_k+\sigma_0/2,\sigma_k} = \bigcup_{i=1}^{n_k} Z_i,$$

$$\mu_r(\delta F_k^1) \sim J_{1-\sigma_k+\sigma_0/2,\sigma_k} \subset \mu_r(\delta F_k) \sim J_{1-\sigma_k+\sigma_0/2,\sigma_k}. \tag{3.3}$$

Hence, as in the proof of Theorem 1, we have $\{C_\alpha\}$ with

$$\bigcup_\alpha C_\alpha = \bigcup_{i=1}^{n_k} \partial Z_i = \mu_r(\delta F_k^1) \cap L_{\sigma_k-\sigma_0/2} \subset \mu_r(\delta F_k) \cap L_{\sigma_k-\sigma_0/2},$$

and the corresponding $\{Y_\alpha\}$ with $Y_\alpha \subset L_{\sigma_k-\sigma_0/2}$ and $Y_\alpha \approx D$. The next step involves a slight perturbation of $\bigcup_{i=1}^{n_k} Z_i$, holding the set $\mu_r(\delta F_k^1) \sim J_{1-\sigma_k+\sigma_0/2,\sigma_k}$ fixed, and taking points of $\bigcup_{i=1}^{n_k} Z_i$ into $\mu_r(M) \sim \overline{J}_{1-\sigma_k+\sigma_0/2,\sigma_k}$ in such a way that each Z_i is taken closely to some Y_α. In this way we can obtain $F_k^2 \in \Phi$ such that

(3.4) $\mu_r(\delta F_k^2) \cap U^3(1-\sigma_k+\sigma_0/2) \subset K_{1-\sigma_k+\sigma_0/2,\sigma_k} \sim \partial K_{1-\sigma_k+\sigma_0/2,\sigma_k}$,

$$H^2(\mu_r(\delta F_k^2) \sim \mu_r(\delta F_k)) \le H^2\left(\bigcup_\alpha Y_\alpha\right) + \delta, \tag{3.5}$$

$$H^2(\mu_r(\delta F_k^2)) \le H^2(\mu_r(\delta F_k)) + H^2\left(\bigcup_\alpha Y_\alpha\right) + \delta, \tag{3.6}$$

for any preassigned $\delta > 0$. Since $\bigcup_\alpha \partial Y_\alpha \subset \bigcup_\alpha C_\alpha = \mu_r(\delta F_k^1) \cap L_{\sigma_k-\sigma_0/2}$ we have by (3.2) and (3.3),

$$H^2\left(\bigcup_\alpha Y_\alpha\right) \le \sum_\alpha a_p(H^1(\partial Y_\alpha))^2 \le a_p\eta^2, \tag{3.7}$$

where a_p depends on the point p and is finite because, for fixed $\sigma_0 > 0$, $L_{\sigma_k-\sigma_0/2}$ cannot have an arbitrarily sharp vertex. Thus, taking δ and η arbitrarily small, we deduce from (3.6) and (3.7) that

$$\lim_{k\to\infty} H^2(\mu_r(\delta F_k^2)) \le \lim_{k\to\infty} H^2(\mu_r(\delta F_k)).$$

It follows from (3.5) that $\lim_{k\to\infty} |\delta F_k^2| = \lim_{k\to\infty} |\delta F_k|$. This proves Lemma 6.

Now choose ρ_k with $7/8 \le \rho_k \le 1-\sigma_k$ (assuming $0 < \sigma_0 < 1/8$) such that $\mu_r(\delta F_k^2)$ is transversal to $\partial B^3(0,\rho_k)$. Note that by (3.4)

$$\mu_r(\delta F_k^2) \cap \partial B^3(0,\rho_k) = \mu_r(\delta F_k^2) \cap \partial B^3(0,\rho_k) \cap K_{1,\sigma_k}.$$

Hence $\mu_r(\delta F_k^2) \cap \partial B^3(0,\rho_k) = \bigcup_{i=1}^{l_k} C_k^i$, where each C_k^i is a compound Jordan curve on $\partial B^3(0,\rho_k)$. Let Y_k^i be the open disk in $\partial B^3(0,\rho_k)$ corresponding to C_k^i as defined in the proof of Theorem 1. Assuming σ_0 is taken small enough, we can deduce that if C_k^i is null-homotopic in $\partial B^3(0,\rho_k) \cap K_{1,\sigma_k}$, then

$$Y_k^i \subset \partial B^3(0,\rho_k) \cap K_{1,\sigma_k}. \tag{3.8}$$

Let $\{Q_k^1, \cdots, Q_k^{a_k}\}$ be the set of all components of $\mu_r(\delta F_k^2) \cap B^3(0,\rho_k)$. Then each component Q_k^j falls on one of the following two cases:

Case I. All components of $Q_k^j \cap \partial B^3(0,\rho_k)$ are null-homotopic in $\partial B^3(0,\rho_k) \cap K_{1,\sigma_k}$.

Case II. A component of $Q_k^j \cap \partial B^3(0,\rho_k)$ is not null-homotopic.

Now we claim the following.

Lemma 7. *Every component of $\mu_r(\delta F_k^2) \cap B^3(0,\rho_k)$ of Case* I *can be discarded to produce another minimizing sequence $\{\delta F_k^3\}$ with the same varifold* *limit* Δ. (*That is, every filigree of* $\mu_r(\delta F_k^2)$ *passing through* K_{ρ_k,σ_k} *can be cut off without changing varifold limit.*)

Proof of Lemma 7. Notice that if C_k^i is null-homotopic in $\partial B^3(0,\rho_k) \cap K_{1,\sigma_k}$ and C_k^j is not null-homotopic, then Y_k^j cannot be a subset of Y_k^i. Moreover if $C_k^i \subset Q_k^{\bar{i}} \cap \partial B^3(0,\rho_k)$ and $C_k^j \subset Q_k^{\bar{j}} \cap \partial B^3(0,\rho_k)$, where $Q_k^{\bar{i}}$ is the component of Case I and $Q_k^{\bar{j}}$ is that of Case II, then Y_k^j cannot be a subset of Y_k^i. Therefore we can find a suitable order $C_k^1 < C_k^2 < \cdots < C_k^{l_k}$, and renumber $\{Q_k^j\}_{1\le j\le a_k}$ in such a way that

(i) $\bigcup_{i=1}^{L_k} C_k^i = \bigcup_{j=1}^{A_k} Q_k^j \cap \partial B^3(0,\rho_k)$, where $\{Q_k^1,\cdots,Q_k^{A_k}\}$ is the set of all Q_k^j's of Case I;

(ii) $\bigcup_{i=L_k+1}^{l_k} C_k^i = \bigcup_{j=A_k+1}^{a_k} Q_k^j \cap \partial B^3(0,\rho_k)$, where $\{Q_k^{A_k+1},\cdots,Q_k^{a_k}\}$ is the set of all Q_k^j's of Case II.

Similarly if we assume $\mu_r(\delta F_k^2)$ is transversal to $\partial B^3(0,t)$ and let

$$\mu_r(\delta F_k^2) \cap \partial B^3(0,t) = \bigcup_{i=1}^{l_k^t} C_k^{i,t}, \qquad 0 < t \le \rho_k,$$

then we can find a suitable order $C_k^{1,t} < \cdots < C_k^{l_k^t,t}$ such that

$$\bigcup_{i=1}^{L_k^t} C_k^{i,t} = \bigcup_{j=1}^{A_k} Q_k^j \cap \partial B^3(0,t), \qquad \bigcup_{i=L_k^t+1}^{l_k^t} C_k^{i,t} = \bigcup_{j=A_k+1}^{a_k} Q_k^j \cap \partial B^3(0,t).$$

Obviously $C_k^{i,\rho_k} = C_k^i$, $l_k^{\rho_k} = l_k$, and $L_k^{\rho_k} = L_k$.

We then apply the process used in the proof of Theorem 1 (with $\mu_r(\delta F_k^2)$ in place of δF and $U^3(0,t)$ in place of U, and so on) only for $\{C_k^{1,t},\cdots,C_k^{L_k^t,t}\}$ (i.e., cutting off the components Q_k^j of Case I only) until we get $\mu_r(\delta F_k^2)$, $\mu_r(\delta F_{k,1}^{2,t}),\cdots,\mu_r(\delta F_{k,L(t)}^{2,t})$, and $U^3(0,t) \supset U_{k,1}^t \supset \cdots \supset U_{k,L(t)}^t$ such that $L(t) \le L_k^t$,

$$\mu_r(\delta F_{k,L(t)}^{2,t}) \sim U^3(0,t) \subset \mu_r(\delta F_k^2) \sim U^3(0,t), \tag{3.9}$$

$$\mu_r(\delta F_{k,L(t)}^{2,t}) \cap U_{k,L(t)}^t = \bigcup_{j=A_k+1}^{a_k} Q_k^j \cap U^3(0,t), \tag{3.10}$$

$$\mu_r(\delta F_{k,L(t)}^{2,t}) \cap (U^3(0,t) \sim U_{k,L(t)}^t) = \bigcup_{\alpha=1}^{\overline{L}(t)} \bigcup_{\beta=1}^{m_\alpha} Z_{k,\beta}^{\alpha,t}, \tag{3.11}$$

where $\overline{L}(t) \le L(t)$, $Z_{k,\beta}^{\alpha,t}$ is an area minimizing surface homeomorphic to D, and

$$\bigcup_{\beta=1}^{m_\alpha} \partial Z_{k,\beta}^{\alpha,t} = C_k^{n_\alpha,t} \quad \text{for some } 1 \le n_\alpha \le L_k^t.$$

By Lemma 6 we have

$$H^2(\delta F_k^2) \leq H^2(\delta G) + \varepsilon_k, \quad \text{for any } G \in \Phi, \tag{3.12}$$

where $\varepsilon_k \to 0$ as $k \to \infty$. Thus

$$\begin{aligned} &H^2(\mu_r(\delta F_k^2) \cap B^3(0,t)) + H^2(\mu_r(\delta F_k^2) \sim B^3(0,t)) \\ &\quad \leq H^2(\mu_r(\delta F^{2,t}_{k,L(t)}) \cap B^3(0,t)) + H^2(\mu_r(\delta F^{2,t}_{k,L(t)}) \sim B^3(0,t)) + r^2\varepsilon_k, \end{aligned}$$

which together with (3.9) implies

$$H^2(\mu_r(\delta F_k^2) \cap B^3(0,t)) = H^2(\mu_r(\delta F^{2,t}_{k,L(t)}) \cap B^3(0,t)) + r^2\varepsilon_k.$$

Hence by (3.10) and (3.11) we have

$$\begin{aligned} &\sum_{j=1}^{A_k} H^2(Q_k^j \cap B^3(0,t)) + \sum_{j=A_k+1}^{a_k} H^2(Q_k^j \cap B^3(0,t)) \\ &\qquad \leq \sum_{\alpha=1}^{\overline{L}(t)} \sum_{\beta=1}^{m_\alpha} H^2(Z^{\alpha,t}_{k,\beta}) + \sum_{j=A_k+1}^{a_k} H^2(Q_k^j \cap B^3(0,t)) + r^2\varepsilon_k, \end{aligned}$$

or

$$\sum_{j=1}^{A_k} H^2(Q_k^j \cap B^3(0,t)) \leq \sum_{\alpha=1}^{\overline{L}(t)} \sum_{\beta=1}^{m_\alpha} H^2(Z^{\alpha,t}_{k,\beta}) + r^2\varepsilon_k. \tag{3.13}$$

Next we claim that $|\bigcup_{j=i}^{A_k} Q_k^j \cap B^3(0,3/4)|$ vanishes as $k \to \infty$. Note first that there are connected subsets $R_k^1, \cdots, R_k^{\overline{L}(\rho_k)}$ of $\mu_r(\delta F_k^2)$ such that

$$\bigcup_{\alpha=1}^{\overline{L}(\rho_k)} R_k^\alpha \cap B^3(0,\rho_k) = \bigcup_{j=1}^{A_k} Q_k^j \cap B^3(0,\rho_k), \tag{3.14}$$

and $\partial R_k^\alpha = \bigcup_{\beta=1}^{m_\alpha} \partial Z^{\alpha,\rho_k}_{k,\beta}$ $(= C_k^{n_\alpha})$.

To justify the above claim we show that there exist nonnegative numbers $\varepsilon_k^1, \cdots, \varepsilon_k^{\overline{L}(\rho_k)}$ with

$$\sum_{\alpha=1}^{\overline{L}(\rho_k)} \varepsilon_k^\alpha \leq r^2\varepsilon_k, \tag{3.15}$$

such that

$$H^2(R_k^\alpha \cap V) \leq H^2(E) + \varepsilon_k^\alpha, \tag{3.16}$$

whenever $V \subset M$ and $E \subset \partial V$ are such that

$$\begin{aligned} &V \approx B^\circ, \partial V \cap R_k^\alpha \text{ is a nonempty compound Jordan curve in } \overline{E}, \\ &V \cap Y_k^{n_\alpha} = \varnothing, \text{ and } (\mu_r(\delta F_k^2) \sim (R_k^\alpha \cap V)) \cup E = \mu_r(\delta G) \\ &\qquad\qquad \text{for some } G \in \Phi. \end{aligned} \tag{3.17}$$

Suppose (3.16) fails; then we must have

$$\sup\{H^2(R_k^\alpha \cap V) - H^2(E)\} = \theta_k^\alpha,$$

where $\alpha = 1, \cdots, \overline{L}(\rho_k)$, $\sum_{\alpha=1}^{\overline{L}(\rho_k)} \theta_k^\alpha > r^2\varepsilon_k$, and the supremum is taken over all V and E satisfying (3.17). Choose $V_1, \cdots, V_{\overline{L}(\rho_k)}, E_1, \cdots, E_{\overline{L}(\rho_k)}$ with

$$H^2(R_k^\alpha \cap V_\alpha) - H^2(E_\alpha) > \overline{\theta}_k^\alpha, \tag{3.18}$$

$$\sum_{\alpha=1}^{\overline{L}(\rho_k)} \overline{\theta}_k^\alpha > r^2\varepsilon_k, \tag{3.19}$$

and define $R_k^{1,\alpha} = (R_k^\alpha \sim V_\alpha) \cup E_\alpha$. Then (3.18) implies

$$H^2(R_k^\alpha) - H^2(R_k^{1,\alpha}) > \overline{\theta}_k^\alpha. \tag{3.20}$$

Now we assert $\{R_k^{1,\alpha}\}_{1\le\alpha\le\overline{L}(\rho_k)}$ can be replaced by a pairwise disjoint family satisfying (3.20). The proof of this assertion is by induction on α. The result is trivial if $\overline{L}(\rho_k) = 1$. Hence take $\overline{L}(\rho_k) \geq 2$ and assume that $\{R_k^{1,\alpha}\}_{1\le\alpha\le\overline{L}(\rho_k)-1}$ is pairwise disjoint. Let $R_k^{2,\alpha}$ be the subset of $R_k^{1,\alpha}$ homeomorphic to $\overline{D}$ with $\partial R_k^{2,\alpha} = \partial Y_k^{n_\alpha}$. Applying Lemma 2 of [9] to $\{R_k^{2,\alpha}\}_{1\le\alpha\le\overline{L}(\rho_k)}$ we get a pairwise disjoint family $\{R_k^{3,\alpha}\}_{1\le\alpha\le\overline{L}(\rho_k)}$ with $\partial R_k^{3,\alpha} = \partial R_k^{2,\alpha}$ and $H^2(R_k^{3,\alpha}) \leq H^2(R_k^{2,\alpha})$. It is obvious from the proof of this lemma that $\{R_k^{3,\alpha}\}_{1\le\alpha\le\overline{L}(\rho_k)}$ gives rise to $\{R_k^{4,\alpha}\}_{1\le\alpha\le\overline{L}(\rho_k)}$ which is pairwise disjoint and satisfies

$$H^2(R_k^\alpha) - H^2(R_k^{4,\alpha}) > \overline{\theta}_k^\alpha, \tag{3.21}$$

thereby proving our assertion.

Replacing $\{R_k^\alpha\}_{1\le\alpha\le\overline{L}(\rho_k)}$ by $\{R_k^{4,\alpha}\}_{1\le\alpha\le\overline{L}(\rho_k)}$, we obtain a new fundamental domain $G \in \Phi$; that is,

$$\left(\mu_r(\delta F_k^2) \sim \left(\bigcup_{\alpha=1}^{\overline{L}(\rho_k)} R_k^\alpha\right)\right) \cup \left(\bigcup_{\alpha=1}^{\overline{L}(\rho_k)} R_k^{4,\alpha}\right) = \mu_r(\delta G)$$

for some $G \in \Phi$. Therefore

$$H^2(\mu_r(\delta F_k^2)) - H^2(\mu_r(\delta G)) = \sum_{\alpha=1}^{\overline{L}(\rho_k)} [H^2(R_k^\alpha) - H^2(R_k^{4,\alpha})].$$

Hence (3.19) and (3.21) give $H^2(\mu_r(\delta F_k^2)) - H^2(\mu_r(\delta G)) > r^2\varepsilon_k$, contradicting (3.12). Thus (3.16) follows.

Now let us justify the above mentioned claim. If we define $f_k^\alpha(t) = H^2(R_k^\alpha \cap B^3(0,t)) - \varepsilon_k^\alpha$, $t \in [t_k^\alpha, \rho_k]$, $t_k^\alpha \equiv \inf\{t\colon f_k^\alpha(t) > 0\}$, then (3.16) with $\overline{E} = \bigcup_{\beta=1}^{m_\alpha} \overline{Z}_{k,\beta}^{\alpha,t}$ implies $f_k^\alpha(t) \le (2\pi)^{-1}(\frac{d}{dt} f_k^\alpha(t))^2$ because

$$H^2\left(\bigcup_{\beta=1}^{m_\alpha} Z_{k,\beta}^{\alpha,t}\right) \le H^2(Y_k^{n_\alpha,t}) \le \frac{1}{2\pi}(H^1(\partial Y_k^{n_\alpha,t}))^2 \le \frac{1}{2\pi}(H^1(C_k^{n_\alpha,t}))^2$$
$$\le \frac{1}{2\pi}(H^1(R_k^\alpha \cap \partial B^3(0,t)))^2 \le \frac{1}{2\pi}\left(\frac{d}{dt} f_k^\alpha(t)\right)^2.$$

Proceeding as in the proof of Lemma 3 (Filigree lemma) we deduce that

$$H^2(R_k^\alpha \cap B^3(0,t)) \le \varepsilon_k^\alpha,$$
$$\text{whenever } t \le \rho_k - \sqrt{2/\pi}\sqrt{H^2(R_k^\alpha \cap B^3(0,\rho_k))}. \tag{3.22}$$

Now by using (3.8) and (3.16) we have

$$H^2(R_k^\alpha \cap B^3(0,\rho_k)) \le H^2(Y_k^{n_\alpha}) + \varepsilon_k^\alpha \le H^2(\partial B^3(0,\rho_k) \cap K_{1,\sigma_k}) + \varepsilon_k^\alpha.$$

Since $H^2(\partial B^3(0,\rho_k) \cap K_{1,\sigma_k}) \le 2\sigma_k \Theta^2(\mu,p) 2\pi\rho_k$, we deduce

$$H^2(R_k^\alpha \cap B^3(0,\rho_k)) \le 4\pi\rho_k\sigma_k\Theta^2(\mu,p) + \varepsilon_k^\alpha \le 5\pi\sigma_0\Theta^2(\mu,p)$$

for sufficiently large k (remember $\frac{3}{4}\sigma_0 < \sigma_k < \sigma_0$ and $\rho_k \le 1 - \sigma_k$). Thus (3.22) implies $H^2(R_k^\alpha \cap B^3(0,3/4)) \le \varepsilon_k^\alpha$ provided σ_0 is sufficiently small. Therefore from (3.14) and (3.15) we have

$$H^2\left(\bigcup_{j=1}^{A_k} Q_k^j \cap B^3(0,3/4)\right) = \sum_{\alpha=1}^{\overline{L}(\rho_k)} H^2(R_k^\alpha \cap B^3(0,3/4)) \le r^2\varepsilon_k.$$

Hence $|\bigcup_{j=1}^{A_k} Q_k^j \cap B^3(0,3/4)|$ vanishes as $k \to \infty$, provided σ_0 is sufficiently small, as claimed above.

Perturbing $\mu_r(\delta F_k^2)$ slightly, if necessary, we can assume that $\mu_r(\delta F_k^2)$ is transversal to $\partial B^3(0,3/4)$. Choose a point $q \in J_{3/4,1/4}$ $(= U^3(0,3/4) \sim K(3/4,1/4))$ and let π_q be the radial projection map from q onto $\partial B^3(0,3/4)$. Then clearly we have for some $c_q < \infty$

$$H^2\left(\pi_q\left(\bigcup_{j=1}^{A_k} Q_k^j \cap B^3(0,3/4)\right)\right) \le c_q H^2\left(\bigcup_{j=1}^{A_k} Q_k^j \cap B^3(0,3/4)\right).$$

Since

$$H^2\left(\bigcup_{\alpha=1}^{\overline{L}(3/4)} \bigcup_{\beta=1}^{m_\alpha} Z_{k,\beta}^{\alpha,3/4}\right) \le H^2\left(\bigcup_{\alpha=1}^{\overline{L}(3/4)} Y_k^{n_\alpha,3/4}\right) \quad \text{and}$$

$$\bigcup_{\alpha=1}^{\overline{L}(3/4)} Y_k^{n_\alpha,3/4} \subset \pi_q\left(\bigcup_{j=1}^{A_k} Q_k^j \cap B^3(0,3/4)\right),$$

we have, by the above inequality,

$$\lim_{k\to\infty} H^2\left(\bigcup_{\alpha=1}^{\overline{L}(3/4)}\bigcup_{\beta=1}^{m_\alpha} Z_{k,\beta}^{\alpha,3/4}\right) = 0.$$

It follows from (3.9), (3.10), and (3.11) that

$$\lim_{k\to\infty} |\delta F_{k,L(3/4)}^{2,3/4}| = \lim_{k\to\infty} |\delta F_k^2|.$$

Define $F_k^3 = F_{k,L(3/4)}^{2,3/4}$, which completes the proof of Lemma 7.

Finally we cut off the filigrees arising from components Q_k^j's of Case II in the following lemma.

Lemma 8. *$\{F_k^3\}$ can be replaced by another minimizing sequence $\{F_k^4\}$ with the property that $\lim_{k\to\alpha} |\delta F_k^4| = \Delta$, and $\mu_r(F_k^4) \cap \tilde{B}^3(0,3/4)$ lies in a bounded set in $\mu_r(\tilde{M})$. (That is, every filigree of $\partial(\mu_r(F_k^3))$ in $\mu_r(\tilde{M})$ passing through $\tilde{K}_{3/4,\sigma_k}$ can be cut off.)*

Proof of Lemma 8. In this lemma we assume without loss of generality that no component of $\bigcup_{j=A_k+1}^{a_k} Q_k^j \cap \partial B^3(0,t)$, $0 < t \le \rho_k$, is null-homotopic in $\partial B^3(0,t) \cap K_{1,\sigma_k}$ since such bad components can be eliminated by cutting (Lemma 2) and slight perturbation, thereby decreasing the area of $\mu_r(\delta F_k^3)$.

By Corollary 2, the slice of μ in $\partial B^3(0,1)$, as defined in [3, I.3(3)] and denoted by $\langle\mu, \text{dist}, 1\rangle$, is stationary in $\partial B^3(0,1)$ and hence by the structure theorem [2] the number of components of $\partial B^3(0,1) \sim \text{spt}\,\|\mu\|$ is finite, say $c_p < \infty$. Therefore $\partial B^3(0,t) \sim K_{1,\sigma_k}$ has at most c_p components for all $t \le \rho_k$. Let $L_1, \cdots, L_{d_k}$, $d_k \le c_p$, be all components of $\partial B^3(0,\rho_k) \sim K_{1,\sigma_k}$, and $L_{k,1}^j, \cdots, L_{k,b(j,k)}^j$, all components of $\partial B^3(0,\rho_k) \sim Q_k^j$. Then each L_m, $1 \le m \le d_k$, is a subset of $L_{k,l}^j$ for some $1 \le l \le b(j,k)$, and each $\partial L_{k,l}^j$ is an individual Jordan curve of the compound Jordan curve $Q_k^j \cap \partial B^3(0,\rho_k)$. Renumbering $L_{k,1}^j, \cdots, L_{k,b(j,k)}^j$, if necessary, we can assume that for $\overline{b}(j,k)+1 \le l \le b(j,k)$, $L_{k,l}^j \subset \partial B^3(0,\rho_k) \cap K_{1,\sigma_k}$ and for $1 \le l \le \overline{b}(j,k)$, $L_{k,l}^j \cap (\partial B^3(0,\rho_k) \sim K_{1,\sigma_k}) \ne \varnothing$. Hence $\overline{b}(j,k) \le d_k \le c_p$. Let $O_{k,1}^j, \cdots, O_{k,\hat{b}(j,k)}^j$ be the set of all components of $B^3(0,\rho_k) \sim Q_k^j$, $A_k + 1 \le j \le a_k$, which are numbered in such a way that $\overline{b}(j,k) \le \hat{b}(j,k) \le b(j,k)$ and $\partial B^3(0,\rho_k) \cap O_{k,l}^j \supset L_{k,l}^j$ for $1 \le l \le \overline{b}(j,k)$. We then note that, for each $\overline{b}(j,k)+1 \le l \le \hat{b}(j,k)$, $Q_k^j \cap \overline{O}_{k,l}^j$ is the image under the projection map p of a filigree of $\partial(\mu_r(\delta F_k^3))$ in $\mu_r(\tilde{M})$ passing through $\tilde{K}_{\rho_k,\sigma_k}$. We are to cut off these filigrees by attaching each $O^j_{k,l}$, $\bar{b}(j,k)+1 \le l \le \hat{b}(j,k)$, to an appropriate $O^j_{k,l}$, $1 \le l \le \bar{b}(j,k)$. To do so, define $O^j_k = \bigcup_{l=\bar{b}(j,k)+1}^{\hat{b}(j,k)} \overline{O}^j_{k,l}$.

In our cutting and pasting arguments so far, cutting has taken place inside a set $X \subset\subset Y$ whose boundary ∂X is the outermost Jordan curve of $\mu_r(\delta F) \cap Y$ (Lemma 2). This time, however, applying the methods of Remark 1, we cut $\mu_r(p(F^3_k))$ along $\partial B^3(0,t) \cap O^j_k$, $0 < t \le \rho_k$. Let I^t be the family of all components of $(\partial B^3(0,t) \sim Q^j_k) \cap O^j_k$. First, suppose $X \in I^t$ and note that $\mu_r(p(F^3_k)) \sim X$ is the union of two disjoint open sets V_1 and V_2 which are homeomorphic to B°. Hence we paste V_1 to V_2 along a subset of $\partial V_1 \cap \partial V_2 \sim X$ which is a common face of V_1 and V_2 other than X, to get a new fundamental domain homeomorphic to B° (Lemma 5 assures that this pasting does not change the topology of fundamental domain).

Secondly, more generally, $\mu_r(p(F^3_k)) \sim \bigcup_{X_i \in I^t_m} X_i$ is a disjoint union of open balls for any subfamily I^t_m of I^t, and hence it can be pasted appropriately to give a new fundamental domain $F^{3,t}_{k,j,m} \in \Phi$. Here we can arrange this cutting and pasting process in such a way that the resulting fundamental domain $F^{3,t}_{k,j,m} \in \Phi$ satisfies the following properties:

$$\mu_r(\delta F^{3,t}_{k,j,m} \sim \mu_r(\delta F^3_k) \subset \bigcup_{X_i \in I^t_m} X_i, \tag{3.23}$$

$$\mu_r(\delta F^3_k) \sim \mu_r(\delta F^{3,t}_{k,j,m}) \supset \overline{O}^j_{k,m} \cap \partial O^j_k \cap U^3(0,t), 1 \le m \le \bar{b}(j,k). \tag{3.24}$$

Furthermore it is not difficult to arrange the above cutting and pasting process so that

$$\bigcup_{m=1}^{\bar{b}(j,k)} [\mu_r(\delta F^3_k) \sim \mu_r(\delta F^{3,t}_{k,j,m})] \supset Q^j_k \cap O^j_k \cap U^3(0,t). \tag{3.25}$$

Now Lemma 7 implies that $H^2(\delta F^3_k) \le H^2(\delta G) + \varepsilon_k$ for any $G \in \Phi$, where $\varepsilon_k \to 0$ as $k \to \infty$, so that $H^2(\mu_r(\delta F^3_k)) \le H^2(\mu_r(\delta F^{3,t}_{k,j,m})) + r^2\varepsilon_k$. Thus

$$H^2(\mu_r(\delta F^3_k) \sim \mu_r(\delta F^{3,t}_{k,j,m})) \le H^2(\mu_r(\delta F^{3,t}_{k,j,m}) \sim \mu_r(\delta F^3_k)) + r^2\varepsilon_k.$$

Hence (3.23) and (3.25) give

$$H^2(Q^j_k \cap O^j_k \cap U^3(0,t)) \le \bar{b}(j,k)\left(H^2\left(\bigcup_{X_i \in I^t_m} X_i\right) + r^2\varepsilon_k\right). \tag{3.26}$$

Since $\bar{b}(j,k) \le c_p$ we can replace $\bar{b}(j,k)$ by c_p in (3.26).

Define

$$f^j_k(t) = H^2(Q^j_k \cap O^j_k \cap U^3(0,t)) - c_p r^2 \varepsilon_k, t \in [t^j_k, \rho_k], t^j_k = \inf\{t \colon f^j_k(t) > 0\}.$$

Then by (3.26) we have $f_k^j(t) \le c_p H^2(\bigcup_{X_i \in I_m^t} X_i)$. From the isoperimetric inequality on $\partial B^3(0,t)$ and the co-area formula it follows that

$$f_k^j(t) \le \frac{c_p}{2\pi}\left(\frac{d}{dt} f_k^j(t)\right)^2 \quad \text{a.e. } t \in [t_k^j, \rho_k].$$

By applying the same method as in Lemma 3 (Filigree lemma) we obtain

$$H^2(Q_k^j \cap O_k^j \cap U^3(0,t)) \le c_p r^2 \varepsilon_k$$

whenever

$$t \le \rho_k - \sqrt{2c_p/\pi}\sqrt{H^2(Q_k^j \cap O_k^j \cap U^3(0,\rho_k)}.$$

Now using (3.26) we get

$$H^2(Q_k^j \cap O_k^j \cap U^3(0,\rho_k)) \le c_p H^2(K_{1,\sigma_k} \cap \partial B^3(0,\rho_k)) + c_p r^2 \varepsilon_k.$$

Hence $H^2(Q_k^j \cap O_k^j \cap U^3(0,\rho_k)) \le 4\pi c_p \rho_k \sigma_k \Theta^2(\mu,p) + c_p r^2 \varepsilon_k$. Therefore for sufficiently large k, we have

$$H^2(Q_k^j \cap O_k^j \cap U^3(0,\rho_k)) \le 5\pi c_p \sigma_0 \Theta^2(\mu,p).$$

Thus we obtain $H^2(Q_k^j \cap O_k^j \cap U^3(0,3/4)) \le c_p r^2 \varepsilon_k$, provided σ_0 is sufficiently small. Hence $|Q_k^j \cap O_k^j \cap U^3(0,3/4)|$ vanishes as $k \to \infty$. Then, by using the projection map π_q as in the proof of Lemma 7, we see that

$$H^2(O_k^j \cap \partial B^3(0,3/4)) \to 0 \quad \text{as } k \to \infty. \tag{3.27}$$

By the assumption at the beginning of this proof each component of

$$\left(\bigcup_j O_k^j \cap \partial B^3(0,3/4)\right) \sim \mu_r(\delta F_k^3)$$

is homeomorphic to D. Hence $\mu_r(p(F_k^3)) \sim (\bigcup_j O_k^j \cap \partial B^3(0,3/4)) \approx \bigcup_i V_i$, $V_i \approx B^\circ$. Now, pasting $\bigcup_j O_k^j \cap U^3(0,3/4)$ to $U^3(0,3/4) \sim \bigcup_j O_k^j$ in an appropriate way and performing, if necessary, more pastings inside $\mu_r(M) \sim U^3(0,3/4)$, we can get $F_k^4 \in \Phi$ such that

$$\mu_r(\delta F_k^4) \sim \mu_r(\delta F_k^3) \subset \bigcup_j O_k^j \cap \partial B^3(0,3/4), \tag{3.28}$$

$$\left(\mu_r(\delta F_k^4) \cap B^3\left(0,\frac{3}{4}\right)\right) \cup \left(\bigcup_j O_k^j \cap \partial B^3\left(0,\frac{3}{4}\right)\right) \tag{3.29}$$

encloses no domain in $U^3(0,\frac{3}{4})$. Note however that $(\mu_r(\delta F_k^3) \cap B^3(0,3/4)) \cup (\bigcup_j O_k^j \cap \partial B^3(0,3/4))$ encloses domains which are subsets of $\bigcup_j O_k^j \cap B^3(0,3/4)$.

By (3.27) and (3.28) we have $\lim_{k\to\infty}|\delta F_k^4| = \lim_{k\to\infty}|\delta F_k^3|$. Thus we see that no spike, or filigree, of $\partial(\mu_r(F_k^4))$ in $\mu_r(\tilde{M})$ can come into $\tilde{U}^3(0,3/4)$. Hence $B^3(0,3/4) \sim \mu_r(\delta F_k^4)$ has only finite components, and therefore $\mu_r(F_k^4) \cap \tilde{B}^3(0,3/4)$ lies in a bounded set in $\mu_r(\tilde{M})$. This completes the proof of Lemma 8.

Finally we are in a position to finish the proof of Theorem 2. In Lemma 8 r depends on p, say, $r = r_p$. By compactness of M there exists a set I of a finite number of points in M such that $\{U^3(q, 3/(4r_q))\}_{q\in I}$ covers M. Hence, applying Lemma 8 repeatedly at all points of I, we deduce that $\{F_k^4\}$ must be uniformly bounded in $\tilde{M}$ provided $\bigcap_k F_k^4 \neq \emptyset$.

Now remember that we assumed at the beginning of the proof of Theorem 2 that M is locally isometrically in $\mathbf{R}^3$. However we can easily see that if M is not locally in $\mathbf{R}^3$, all the methods we have used so far are directly applicable to the images under the exponential map (i.e., exponential image of the tangent cone μ, exponential image of $K_{\rho,\sigma}$, etc). Hence the proof is complete.

The following corollary says that Δ $(= \lim_{k\to\infty}|\delta F_k|)$ is an area minimizing integral varifold in M and is regular in a neighborhood of any point of spt $\|\Delta\|$ where there is a varifold tangent with support contained in a plane. (By rectifiability there is such a tangent plane at almost all points of spt $\|\Delta\|$.)

Corollary 3 (*a.e. smoothness*). *If Δ is the varifold limit of a minimizing sequence in Φ_0 (or Φ), and has a varifold tangent μ at $p \in$ spt $\|\Delta\|$ with* spt $\|\mu\| \subset H$, *where H is a plane, then there is an $r > 0$ such that $\|\Delta\| \llcorner B^3(p,r) = \|n|S|\|$, where n is a positive integer and S is a smooth (analytic if the metric of M is analytic) oriented connected minimal surface containing p.*

Proof. Since spt $\|\mu\|$ is a plane we deduce from (3.29) that each component of $\mu_r(\delta F_k^4) \cap B^3(0,3/4)$ is homeomorphic to a disk, and hence each component of $\mu_r(\delta F_k^4) \cap \partial B^3(0,3/4)$ is a circle which is not null-homotopic in $\partial B^3(0,3/4) \cap K_{1,\sigma_k}$. Then by using the arguments in [4, Theorem 2], we conclude the required result.

4. Fundamental domains with least boundary area

Since now we have a uniformly bounded minimizing sequence of fundamental domains, we can show the existence and regularity of fundamental domains of M which minimize boundary area among all fundamental domains in Φ_0.

Given a continuous map g on M we define $\tilde{g}$ to be a map on $\tilde{M}$ satisfying $p\tilde{g} = gp$. Of course $\tilde{g}$ is not unique, but its uniqueness is not necessary in our setting. For a function h on $\tilde{M}$ we define the function h^M on M by

$$h^M(x) = \sum_{y \in p^{-1}(x)} h(y), \qquad x \in M.$$

Thus h^M is well defined only for a restricted family of functions h on $\tilde{M}$. If F is a fundamental domain of M, then $(\chi_F)^M = 1$ almost everywhere in M, where χ_F is the characteristic function of F on $\tilde{M}$. Also if $H^2(\partial F) < \infty$, then we can easily see that χ_F is a BV function on $\tilde{M}$. Since $\partial^* F$ represents "actual" boundary of F, we may think of $|D\chi_F|(\tilde{M})$ as the boundary area of F (recall $|D\chi_F|(\tilde{M}) = H^2(\partial^* F)$).

Let $I = \inf\{H^2(\partial^* F)\colon F \in \Phi_0\}$ and let $\{F_k\}$ be a sequence of fundamental domains in Φ_0 such that $H^2(\partial^* F_k) \to I$. Then by compactness of BV functions [11, 6.3] there are a subsequence $\{\chi_{F'_k}\} \subset \{\chi_{F_k}\}$ and a BV function u on $\tilde{M}$ such that

$$\chi_{F'_k} \to u \quad \text{in } L^1_{\text{loc}}(\tilde{M}), \qquad |Du|(\tilde{M}) \le \liminf |D\chi_{F'_k}|(\tilde{M}) \quad (= I).$$

u is obviously a characteristic function χ_F of some set F on $\tilde{M}$. By Theorem 2 the sequence $\{F'_k\}$ can be assumed to be uniformly bounded in $\tilde{M}$. Hence

$$1 = (\chi_{F'_k})^M \to (\chi_F)^M \quad \text{in } L^1_{\text{loc}}(M).$$

It follows that F itself is a fundamental domain of M and $H^2(\partial^* F) = I$.

Now the following questions about F arise:

(a) Is $p(\partial^* F)$ locally area minimizing under a Lipschitz map on M?

(b) Is F connected?

If both questions are answered affirmatively, then F is the desired fundamental domain with least boundary area.

Question (a). Let k be a C^1 map on $\tilde{M}$ (or M), and h a function on $\tilde{M}$ (or M). Define the function h^k on $\tilde{M}$ (or M), by

$$h^k(x) = \sum_{y \in k^{-1}(x)} \omega(y) h(y),$$

where

$$\omega(y) = \begin{cases} 1 & \text{if } k \text{ is orientation-preserving at } y, \\ -1 & \text{if } k \text{ is orientation-reversing at } y, \\ 0 & \text{if } Jk(y) = 0, \text{ where } Jk \text{ is the Jacobian of } k. \end{cases}$$

Note that even if k is Lipschitz, h^k is defined almost everywhere. Let g be a Lipschitz map on M such that $\{x\colon g(x) \ne x\} \cup g\{x\colon g(x) \ne x\}$ is contained in a small ball in M. Then by the above definition it is not difficult to check that $((\chi_F)^{\tilde{g}})^M = ((\chi_F)^M)^g = 1^g = 1$ a.e. on M, and therefore also to show that there exists a fundamental domain F_g of M such that

$$\operatorname{spt}|D\chi_{F_g}| \subset \operatorname{spt}|D(\chi_F)^{\tilde{g}}|.$$

Since $\operatorname{spt}|D(\chi_F)^{\tilde{g}}| = \tilde{g}(\operatorname{spt}|D\chi_F|) = \tilde{g}(\partial^* F)$, we have

$$H^2(p(\partial^* F_g)) \leq H^2(p(\tilde{g}(\partial^* F))) \quad (= H^2(g(p(\partial^* F))))),$$

or $H^2(p(\partial^* F)) \leq H^2(g(p(\partial^* F)))$. Therefore $p(\partial^* F)$ is locally area minimizing.

Note. J. Taylor's arguments in [13] might be true even if one assumes that all the Lipschitz maps in [13] are nowhere orientation-reversing. In this case the interior of $\tilde{g}(F)$ is clearly a fundamental domain of M.

Question (b). Suppose F is not connected. Then there are two components U, V of F and subsets X, Y of ∂U, ∂V respectively such that $p(U) \cap p(V) = p(X) = p(Y)$ and $H^2(p(X)) > 0$. Hence there must exist a translation τ on $\tilde{M}$ for which $\tau(X) = Y$. It follows that $(F \sim U) \cup \tau(U)$ is a fundamental domain with less boundary area than F. This contradiction proves the connectedness of F.

Let ν be a diffeomorphism from $B^3(p,r) \subset M$, $p \in \delta F$, to $B^3(0,r) \subset \mathbf{R}^3$, and let $\mathbf{F}$ be the measure over $\mathbf{R}^3$ corresponding to H^2 over M under ν, i.e., $\mathbf{F}(\nu(S)) = H^2(S)$ for any H^2 measurable subset S of M. Then $\nu(\delta F)$ $(= \nu(p(\partial^* F)))$ is locally $\mathbf{F}$-minimizing under Lipschitz deformation in the sense that for any Lipschitz map ϕ on M with $\{x\colon \phi(x) \neq x\} \cup \phi\{x\colon \phi(x) \neq x\} \subset B^3(p,r)$ we have

$$\mathbf{F}(\nu(\delta F)) \leq \mathbf{F}(\nu(\phi(\delta F))).$$

On the other hand one can find a function $\xi(r) = Cr^\alpha$ with $0 \leq C < \infty$ and $0 < \alpha < 1/3$ such that

$$H^2(\nu(\delta F \cap W)) \leq (1 + \xi(r)) H^2(\nu(\phi(\delta F \cap W))),$$

where $W = \{x\colon \phi(x) \neq x\}$ and $r = \operatorname{diam}(W \cup \phi(W))$. Hence $\nu(\delta F)$ is $(\mathbf{M}, \xi, \delta)$ minimal as defined in [13, I. (8)].

Thus by [13, II.4, II.6, IV.5, IV.8] we get the following theorem.

Theorem 3. *There exists a fundamental domain $F \in \Phi_0$ with least boundary area among all elements of Φ_0. Moreover,*

(i) $\delta F = R(\delta F) \cup \sigma_Y(\delta F) \cup \sigma_T(\delta F)$;

(ii) $\sigma_T(\delta F)$ *consists of isolated points*;

(iii) $\sigma_Y(\delta F)$ *is a one-dimensional $C^{1,\alpha}$ submanifold*;

(iv) $R(\delta F)$ *is a smooth minimal surface*;

(v) *for every* $p \in \sigma_Y(\delta F)$ (*respectively* $\sigma_T(\delta F)$) *there is a neighborhood* N *of* p *and a* $C^{1,\alpha/2}$ *diffeomorphism* $f\colon B \to N$ *such that* $\delta F \cap N = f(Y)$ (*respectively* $f(T)$).

Remark 2. Suppose $\hat{M}$ is a covering space of M which is not necessarily the universal covering space $\tilde{M}$ of M. We define a "fundamental domain" F of M in $\hat{M}$ as we did in $\tilde{M}$. Then we can again conclude that there exists a fundamental domain F in $\hat{M}$ with least boundary area and satisfying the regularity results (i)–(v) of Theorem 3. Obviously the proof of Theorem 3 remains valid for fundamental domains in $\hat{M}$.

As for the two-dimensional compact Riemannian manifold M^2, we get the following proposition concerning the existence, regularity, and topology of fundamental domains with least boundary length.

Proposition. *There exists a fundamental domain* F *of* M^2 *which minimizes boundary length among all fundamental domains of arbitrary topological type with the properties that*

(i) $\overline{F}$ *is homeomorphic to a closed disk*;

(ii) F *is a polygon such that the edges of* F *are geodesic segments in* $\tilde{M}$, *interior angles of vertices of* F *are* $120°$, *and if* $\chi(M^2) \leq 0$ *then the number of vertices of* F (= *the number of edges of* F) *is equal to* $6 - 6\chi(M^2)$.

Proof. Note that a minimizing sequence of connected fundamental domains of M^2 must be uniformly bounded if they have nonempty intersection. Therefore we can proceed as above, using characteristic functions of fundamental domains, to conclude that there exists a connected fundamental domain F with least boundary area. Suppose $\overline{F}$ is multiply connected, and $J_1 \approx D$ is a component of $\tilde{M}^2 \sim \overline{F}$ with $H^2(J_1) < \infty$. Then there exists a translation τ_1 on $\tilde{M}^2$ such that $\tau_1(F) \subset J_1$. Since $\tau_1(\overline{F})$ is multiply connected, we have a component $J_2 \approx D$ of $J_1 \sim \tau_1(\overline{F})$ and a translation τ_2 such that $\tau_2(F) \subset J_2$. Continuing this process we can get translations $\tau_1, \tau_2, \tau_3, \cdots$ such that the fundamental domains $\tau_1(F)$, $\tau_2(F)$, $\tau_3(F), \cdots$ are subsets of J_1 and pairwise disjoint. This is not possible since $H^2(J_1) < \infty$. Therefore $\overline{F} \approx \overline{D}$.

The first part of (ii) follows from the fact that δF is locally area minimizing under a Lipschitz map. Now we note that a straight line and Y^1 are the only area minimizing (under the Lipschitz map) 1-varifolds up to rotation (see [13, II.3.]). Hence we deduce the second part of (ii). For the third part of (ii) we recall the Gauss-Bonnet formula,

$$\int_F K\,dA = -\int_{\partial F} \kappa_g\,ds + \sum_{i=1}^{n} (\alpha_i - \pi) + 2\pi,$$

where K is the Gaussian curvature of $\hat{M}$ (or M), κ_g is the signed geodesic curvature of ∂F, and $\alpha_1, \cdots, \alpha_n$ are the interior angles of vertices of F.

Since $\int_F K\,dA = \int_{M^2} K\,dA = 2\pi\chi(M^2)$ and $\kappa_g = 0$, we obtain $2\pi\chi(M^2) = -n\pi/3 + 2\pi$, which gives the last part of (ii).

5. Regularity of singular set

In this section we show the existence and regularity of a fundamental domain in Φ with least boundary area. Without loss of generality we assume, in Lemma 9 and Corollary 4, that M is locally isometrically in $\mathbf{R}^3$ and that for any $p \in \operatorname{spt}\|\Delta\|$ we choose p to be the origin in $\mathbf{R}^3$.

So far we have not ruled out the possibilities that $\mu_r(\delta F_k^4) \cap B^3(0, 3/4)$ might have more than one component, and that the tangent cone μ of Δ at $p \in \operatorname{spt}\|\Delta\|$ might be other than $|D|$, $|Y|$, or $|T|$. The following lemma rules out these possibilities.

Lemma 9. *For any $p \in \operatorname{spt}\|\Delta\|$ the tangent cone μ of Δ at p is $|D|$, $|Y|$, or $|T|$.*

Proof. Suppose $\mu_r(\delta F_k^4) \cap B^3(0, 3/4)$ has more than one component for sufficiently large k. Then we can find a component Q_k of $\mu_r(\delta F_k^4) \cap B^3(0, 3/4)$ for each k and a cone τ such that $\lim_{r\to\infty, k\to\infty} |Q_k| = \tau$, $\operatorname{spt}\|\tau\| \subset \operatorname{spt}\|\mu\|$, and $\tau \neq \mu$. If $\operatorname{spt}\|\tau\|$ were not a plane, then we could construct a diffeomorphism of B fixing ∂B which would decrease not only the mass of τ but also the area of $\mu_r(\delta F_k^4)$ for large r and k, an obvious contradiction. Thus we conclude that if $\mu_r(\delta F_k^4) \cap B^3(0, 3/4)$ has more than one component for large k, then $\mu = m|D|$, $m > 1$.

Next assume that $\mu_r(\delta F_k^4) \cap B^3(0, 3/4)$ is connected for large k. Then we know from the construction of $\mu_r(\delta F_k^4)$ (Lemma 8) that $\mu = |\operatorname{spt}\|\mu\||$. Assume that $\operatorname{spt}\|\mu\| \neq D, Y, T$. First, if $\operatorname{spt}\|\mu\|$ is one of those seven cones which are proven to be not area minimizing under Lipschitz deformation in [13, II.3], then we note that every Lipschitz map ψ constructed in [13, II.3] for each non-area-minimizing cone ν satisfies the property that each component of $U^3(0, 1) \sim \psi(\operatorname{spt}\|\mu\|)$ is homeomorphic to B° and

$$H^2(\psi(\operatorname{spt}\|\nu\|) \cap B^3(0, 1)) < H^2(\operatorname{spt}\|\nu\| \cap B^3(0, 1)).$$

Hence it follows that for large r and k, we can similarly construct the Lipschitz map ψ on $\mu_r(M)$ which leaves $\mu_r(M) \sim B^3(0, 1)$ fixed such that

$$(5.1) \qquad \begin{cases} \mu_r(M) \sim \psi(\mu_r(\delta F_k^4)) \approx B^\circ \quad \text{and} \\ H^2(\psi(\mu_r(\delta F_k^4))) < \mathbf{M}(\mu_{r\#}\Delta) \leq H^2(\mu_r(\delta F_k^4)), \end{cases}$$

an obvious contradiction. Second, if $\operatorname{spt}\|\mu\|$ is different from those cones of [13, II.3], that is, its intersection with $\partial B^3(0, 1)$ is a 1-varifold with multiple points other than triple point (i.e., quadruple point, etc.), then, in view of a

Lipschitz map on $\partial B^3(0,1)$ which squashes a quadruple point to become two triple points and decreases the length of spt $\|\mu\| \cap \partial B^3(0,1)$, we can construct ψ satisfying (5.1) more easily than in the above case. Therefore spt $\|\mu\|$ must be equal to D, Y, or T, and hence, we deduce that $\mu = |D|$, $|Y|$, or $|T|$ in case $\mu_r(\delta F_k^4) \cap B^3(0,3/4)$ is connected for k large.

We now note that by Corollary 3 the density of $R(\Delta)$ is constant in a component of $R(\Delta)$, and the boundary of this component contains the points of $\sigma_Y(\Delta)$. Hence we deduce that if $\mu = m|D|$ along a component of $R(\Delta)$, then m must be equal to 1. Therefore we conclude that $\mu_r(\delta F_k^4) \cap B^3(0,3/4)$ is connected for large k, and $\mu = |D|$, $|Y|$, or $|T|$.

The following corollary and its proof are almost similar to [13, II.6].

Corollary 4 (C^0 *regularity of multiple curve*). (1) $\sigma_T(\Delta)$ *consists of isolated points.*

(2) $\sigma_Y(\Delta)$ *is a one-dimensional* C^0 *submanifold.*

(3) *Suppose* $p \in \sigma_Y(\Delta)$ (*resp.* $\sigma_T(\Delta)$). *Then for some* $r > 0$, $R(\Delta) \cap B^3(p,r)$ *consists of three* (*resp. six*) *components, each of which is a smooth manifold.*

Proof. (1) This conclusion follows from monotonicity (Lemma 4) and the weak convergence to tangent cones as varifolds.

(2) It is not difficult to see that Lemma II.5 of [13] holds in our setting with $\mu_i(\text{spt}\,\|\Delta\|)$ in place of S_i. Hence

$$HD(\sigma_Y(\mu_r\tau_p(\text{spt}\,\|\Delta\| \cap B^3(p,3/4))), \sigma_Y(\phi(Y)))$$

(for some $\phi \in \mathrm{SO}(3)$ depending on p) goes to zero as r goes to ∞ for each fixed $p \in \sigma_Y(\Delta)$ and is uniformly small as a function of p in small compact subsets of $\sigma_Y(\Delta)$ for fixed $r > 0$. Therefore the set $\sigma_Y(\Delta)$ satisfies Reifenberg's condition in §4 of [10], and is hence a one-dimensional C^0 submanifold.

(3) This follows from Corollary 3 and [13, II.6(4)].

An epiperimetric inequality is an inequality which gives us an upper bound to the area of area minimizing surface. This upper bound of area gives us $C^{1,\alpha}$ regularity of area minimizing surface at its singular set. The statement and the proof of epiperimetric inequality basically follow [13]. Before stating the following lemma we should note that the competing surfaces of Δ are not only the images of Δ under diffeomorphisms but also all the varifolds Γ with $\Gamma = \lim_{k\to\infty} |\delta F_k'|$, $F_k' \in \Phi$. We note also that M is no longer assumed locally isometric to $\mathbf{R}^3$. However, using the diffeomorphism from $B^3(p,r) \subset M$ to $B^3(0,r) \subset \mathbf{R}^3$, we equip $M \cap B^3(p,r)$ with the metric $\nu^* g$, where g is the Euclidean metric of $\mathbf{R}^3$.

Lemma 10 (*Epiperimetric inequality for* Δ). *There exist* $\varepsilon > 0$, $\varsigma > 0$, *and* $k > 0$, *such that if*

(i) $p \in \sigma_Y(\Delta)$ (*respectively* $\sigma_T(\Delta)$);
(ii) *for some* $r > 0$,

$$r^{-2}H^2(\operatorname{spt}\|\Delta\| \cap B^3(p,r)) - \pi\Theta^2(\Delta,p) < \varepsilon,$$

and

$$HD(\mu_{1/r}\tau_p(\operatorname{spt}\|\Delta\| \cap B^3(p,r)), \theta Y) < \varsigma$$

(*respectively, replace* θY *by* θT) *for some* $\theta \in \mathrm{SO}(3)$, *then there exists a 2-varifold* Γ *with* $\Gamma = \lim_{k\to\infty} |\delta F_k'|$ *for a sequence* $\{F_k'\}$ *in* Φ *such that*
(1) $\operatorname{spt}\|\Gamma\| \sim B^3(p,r) \subset \operatorname{spt}\|\Delta\| \sim B^3(p,r)$ *and*
(2)

$$\begin{aligned} H^2(\operatorname{spt}\|\Gamma\| \cap B^3(p,r)) \le{}& (1-k/2)(r/2)H^1(\operatorname{spt}\|\Delta\| \cap \partial B^3(p,r)) \\ &+ (k/2)\pi r^2\Theta^2(\Delta,p). \end{aligned}$$

Proof. First, we prove the lemma under the assumption that $\operatorname{spt}\|\Delta\| \cap \partial B^3(p,r)$ consists of a finite number of Lipschitz curves and is homeomorphic to $Y \cap \partial B$ (resp. $T \cap \partial B$). Define $J = \operatorname{spt}\|\Delta\| \cap \partial B^3(p,r)$. Let $\{\phi_n\}$ be a sequence of shrinking diffeomorphisms in $B^3(p,r)$ with $\phi_n(x) = x$, $x \in \partial B^3(p,r)$ for all n such that $\lim_{n\to\infty} \phi_{n\#}\Delta = |p⨳J|$. From [12, Chapter 3] and [13, III.5] one observes that there exist $\varepsilon > 0$, $\varsigma > 0$, and $k > 0$ such that if (i) and (ii) are satisfied then one can find a diffeomorphism ψ with $\{x\colon \psi(x) \ne x\} \cup \psi\{x\colon \psi(x) \ne x\} \subset B^3(p,r)$ such that

$$H^2(\psi(p⨳J)) \le (1-k)(r/2)H^1(J) + k\pi r^2\Theta^2(\Delta,p).$$

In fact, the Lipschitz maps in [12, Chapter 3] can be replaced by diffeomorphisms since the Lipschitz maps there count multiplicity of area. Since by [3, I.2.6] the mapping of the sequence of varifolds $\{\phi_{n\#}\Delta\}$ by ψ is continuous, we have

$$\lim_{n\to\infty} \psi_\#\phi_{n\#}\Delta = \psi_\# \lim_{n\to\infty} \phi_{n\#}\Delta = \psi_\#|p⨳J|.$$

Hence

$$\lim_{n\to\infty} \|(\psi\phi_n)_\#\Delta\|(U^3(p,r)) \le (1-k)(r/2)H^1(J) + k\pi r^2\Theta^2(\Delta,p).$$

Therefore for sufficiently large n,

$$\|(\psi\phi_n)_\#\Delta\|(U^3(p,r)) \le (1-\tfrac{3}{4}k)(r/2)H^1(J) + \tfrac{3}{4}k\pi r^2\Theta^2(\Delta,p).$$

This proves conclusion (2) with k replaced by $\frac{3}{4}k$, $\Gamma = (\psi\phi_n)_\#\Delta$, $F_k' = \tilde{\psi}\tilde{\phi}_n F_k$, and conclusion (1) follows from the choice of ψ and ϕ_n.

Secondly, we prove the lemma under the assumption that $\operatorname{spt}\|\Delta\| \cap \partial B^3(p,r)$ $(= J)$ consists of a finite number of Lipschitz curves. Let K be the subset of J as defined in [13, III.4] such that K is homeomorphic to $Y \cap \partial B$ (resp.

$T \cap \partial B$). Then we get open components U^1, U^2, U^3 (resp. $U^1, \cdots, U^4$) of $U^3(p,r) \sim (p⋇K)$. Now we apply a replacement argument as in Theorem 1 to each U^l and F_k. For this lemma we make one alteration to the proof of Theorem 1. For each U^l we define K_i^l, C_{ij}^l as in Theorem 1. But we let X_{ij1}^l be the component of $\partial U^l \sim C_{ij}^l$ which contains the set $Z^l = \{x \in \partial U^l \colon \operatorname{dist}(x, p⋇K) > \varsigma\}$. Then we define $Y_{ij}^l = \partial U^l \sim \overline{X}_{ij1}^l$ and proceed from here exactly as in Theorem 1. Hence we get $F_k^0 (= F_k)$, F_k^1, F_k^2, F_k^3 (resp. $F_k^0, \cdots, F_k^4) \in \Phi$ with

$$\delta F_k^l \sim U^l \subset \delta F_k^{l-1} \sim U^l, \qquad l = 1,2,3 \text{ (resp. } 1, \cdots, 4). \tag{5.2}$$

Also, if we define $U_\eta^l = \{x \in U^l \colon \operatorname{dist}(x, \partial U^l) < \eta\}$, we can arrange replacement procedure in such a way that $\delta F_k^l \cap U^l \subset U_\eta^l$ for any $\eta > 0$. Hence letting $\eta \to 0$ we obtain

$$\operatorname{spt} \| \lim_{k \to \infty} |\delta F_k^l \cap U^l| \, \| \subset \partial U^l.$$

Moreover we may assume that for each component E of $\delta F_k^l \cap U^l$, $H^2(E)$ is arbitrarily close to $H^2(Z)$, where Z is the subset of ∂U^l with $\partial Z = \partial E$ and $Z \cap Z^l = \varnothing$. Then it follows from (5.2) that since Δ has density 1 almost everywhere the varifold $\lim_{k \to \infty} |\delta F_k^l|$ also has density 1 almost everywhere for each l.

Let $\Delta' = \lim_{k \to \infty} |\delta F_k^3|$ (resp. $\lim_{k\to\infty} |\delta F_k^4|$). Note that the boundary of $\operatorname{spt} \|\Delta'\| \cap U^3(p,r)$ is K. Hence the arguments of the first case is applicable to $\operatorname{spt} \|\Delta'\| \cap U^3(p,r)$ and therefore there exist diffeomorphisms ϕ_n and ψ such that

$$\|(\psi\phi_n)_\# \Delta'\|(U^3(p,r)) \le (1 - \tfrac{3}{4}k)(r/2)H^1(K) + \tfrac{3}{4}k\pi r^2 \Theta^2(\Delta, p). \tag{5.3}$$

On the other hand, [13, III.4(4)] implies

$$H^2(\operatorname{spt} \|\Delta'\| \cap \partial B^3(p,r)) \le 25[H^1(J) - H^1(K)]^2. \tag{5.4}$$

We note however that for any set E in a thin strip with width $2\varsigma r$ we can use an isoperimetric inequality $H^2(E) \le \varsigma r H^1(\partial E)$ instead of $H^2(E) \le [H^1(\partial E)]^2$. Hence (5.4) can be replaced by

$$H^2(\operatorname{spt} \|\Delta'\| \cap \partial B^3(p,r)) \le 5\varsigma r[H^1(J) - H^1(K)]. \tag{5.5}$$

Since $|\operatorname{spt} \|\Delta'\| \, | = \Delta'$, (5.3) and (5.5) yield

$$\begin{aligned} H^2(\operatorname{spt} \|(\psi\phi_n)_\# \Delta'\| \cap B^3(p,r)) &\le (1 - \tfrac{3}{4}k)(r/2)H^1(K) \\ &\quad + 5\varsigma r[H^1(J) - H^1(K)] + \tfrac{3}{4}k\pi r^2 \Theta^2(\Delta, p). \end{aligned} \tag{5.6}$$

Let us assume without loss of generality

$$5\varsigma < (1 - \tfrac{3}{4}k)\tfrac{1}{2}. \tag{5.7}$$

Then (5.6) gives us

$$H^2(\operatorname{spt}\|(\psi\phi_n)_\#\Delta'\|\cap B^3(p,r)) \le (1-\tfrac{3}{4}k)(r/2)H^1(J)+\tfrac{3}{4}k\pi r^2\Theta^2(\Delta,p),$$

which proves conclusion (2) with $F_k' = \tilde{\psi}\tilde{\phi}_n F_k^3$ (resp. $\tilde{\psi}\tilde{\phi}_n F_k^4$). Conclusion (1) follows from (5.2) and the choice of ψ, ϕ_n.

Finally it remains to prove the lemma with no assumption, i.e., without the assumption that J is piecewise Lipschitz. So J needs to be approximated by a finite number of Lipschitz curves. By Corollary 4, there exists $r_q > 0$ for each $q \in \sigma_Y(\Delta)\cap\partial B^3(p,r)$ such that $\sigma_Y(\Delta)\cap\partial B^3(p,r)\cap B^3(q,r_q) = \{q\}$. Let J_q be the component of $\operatorname{spt}\|\Delta\|\cap\partial B^3(p,r)\cap B^3(q,r_q)$ containing q. Then we replace J_q by the union of three geodesic curves $G_q^1\cup G_q^2\cup G_q^3$ in $\partial B^3(p,r)\cap B^3(q,r_q)$ connecting q to the three points $\{q_1,q_2,q_3\} = J_q\cap\partial B^3(q,r_q)$. We assume r_q to be appropriately small so that for sufficiently small $\eta > 0$,

$$H^2\left(\bigcup_q Z_q\right) < \eta, \tag{5.8}$$

where Z_q is the small region on $\partial B^3(p,r)$ enclosed by J_q and $G_q^1\cup G_q^2\cup G_q^3$, and the union is taken over all $q\in\sigma_Y(\Delta)\cap\partial B^3(p,r)$. Let V_q^1, V_q^2, V_q^3 be open components of $p⋇[\partial B^3(p,r)\cap U^3(q,r_q)\sim(G_q^1\cup G_q^2\cup G_q^3)]\sim\partial B^3(p,r)$. By cutting off filigrees of $\{\delta F_k\}$ (applying Lemma 6 repeatedly at suitable points of $\operatorname{spt}\|\Delta\|$) we can find three open balls W_q^1, W_q^2, W_q^3, which are disjoint from δF_k for all k such that

$$W_q^l\cap\partial V_q^l\approx D,\quad W_q^l\cap\partial V_q^l\cap\partial B^3(p,r)\cap\partial B^3(q,r_q)\ne\varnothing,\qquad l=1,2,3.$$

Then we apply the replacement argument of Theorem 1 to each V_q^l and F_k with the same alteration as in the second case: For each component C_{ij} of $\delta F_k\cap\partial V_q^l$, define Y_{ij} to be the subset of $\partial V_q^l\sim\overline{W}_q^l$ such that $Y_{ij}\approx D$ and ∂Y_{ij} is the outermost Jordan curve of C_{ij} in $\partial V_q^l\sim\overline{W}_q^l$. Then, as in Theorem 1, we obtain $F_k^{q,0}(=F_k)$, $F_k^{q,1}$, $F_k^{q,2}$, $F_k^{q,3}$ with

$$\delta F_k^{q,l}\sim V_q^l\subset\delta F_k^{q,l-1}\sim V_q^l,\qquad l=1,2,3.$$

Using the same arguments as in the second case we can assert that

$$\operatorname{spt}\|\lim_{k\to\infty}|\delta F_k^{q,l}\cap V_q^l|\,\|\subset\partial V_q^l$$

and that the density of $\lim_{k\to\infty}|\delta F_k^{q,l}|$ is 1 almost everywhere for each $l = 1,2,3$.

Continue the above process for the sequence of points $\{q,\cdots,P\} = \sigma_Y(\Delta)\cap\partial B^3(p,r)$ until we obtain $\overline{\Delta} = \lim_{k\to\infty}|\delta F_k^{P,3}|$. Then we note that the boundary J' of $\operatorname{spt}\|\overline{\Delta}\|\cap U^3(p,r)$ is piecewise Lipschitz. By finding the subset K'

of J' which is homeomorphic to $Y \cap \partial B$ (resp. $T \cap \partial B$) and using the replacement argument as in the second case above, we can construct a 2-varifold $\hat{\Delta}$ such that $\hat{\Delta} = \lim_{k\to\infty} |\delta \hat{F}_k|$, $\delta \hat{F}_k \in \Phi$, $\hat{\Delta}$ has density 1 almost everywhere,

$$\operatorname{spt}\|\hat{\Delta}\| \sim B^3(p,r) \subset \operatorname{spt}\|\Delta\| \sim B^3(p,r),$$

the boundary of $\operatorname{spt}\|\hat{\Delta}\| \cap U^3(p,r)$ is K', and lastly, from (5.8),

$$H^2(\operatorname{spt}\|\hat{\Delta}\| \cap \partial B^3(p,r)) \leq \eta + 5\varsigma r[H^1(J) - H^1(K')]. \tag{5.9}$$

Hence, as before, we can find diffeomorphisms ϕ_n, ψ such that

$$\|(\psi\phi_n)_\#\hat{\Delta}\|(U^3(p,r)) \leq (1 - \tfrac{3}{4}k)(r/2)H^1(K') + \tfrac{3}{4}k\pi r^2\Theta^2(\Delta,p), \tag{5.10}$$

which together with (5.7), (5.9) implies

$$H^2(\operatorname{spt}\|(\psi\phi_n)_\#\hat{\Delta}\| \cap B^3(p,r)) \leq (1 - k/2)(r/2)H^1(J) + (k/2)\pi r^2\Theta^2(\Delta,p),$$

where η is absorbed by replacing $\frac{3}{4}k$ by $k/2$. This completes the proof of Lemma 10.

Recall the diffeomorphism ν from $B^3(p,r) \subset M$ to $B^3(0,r) \subset \mathbf{R}^3$ which was introduced in §4. If M is given the Euclidean metric which is pulled back by ν, then one can find a function $\xi(r) = Cr^\alpha$ with $0 \leq C < \infty$ and $0 < \alpha < 1/3$ such that

$$H^2(\operatorname{spt}\|\Delta\| \cap W) \leq (1 + \xi(r))H^2(\operatorname{spt}\|\Gamma\| \cap W),$$

where Γ is as defined in Lemma 10, $W = (\operatorname{spt}\|\Delta\| \sim \operatorname{spt}\|\Gamma\|) \cup (\operatorname{spt}\|\Gamma\| \sim \operatorname{spt}\|\Delta\|)$ and $r = \operatorname{diam} W$. This property of Δ is similar to $(\mathbf{M}, \xi, \delta)$ minimality. Therefore we note that the above epiperimetric inequality holds, as [13, III.1], for Δ having this property (not just for area minimizing Δ). The following epiperimetric inequality is similar to [13, III.1] and slightly weaker than Lemma 10.

Lemma 10′. *There exist* $0 < \varepsilon \leq 1/2$, $0 < \varsigma < 1/100$, $0 < k < \alpha/(1+\alpha)$, *and* $1 < l < \infty$, *such that if*

(i) $p \in \sigma_Y(\Delta)$ *(resp.* $\sigma_T(\Delta)$*);*
(ii) *for some* $r > 0$, $r^{-2}H^2(\operatorname{spt}\|\Delta\| \cap B^3(p,r))\exp(\frac{3}{\alpha}\xi(r)) - \pi\Theta^2(\Delta,p) < \varepsilon$, *and* $HD(\mu_{1/r}\tau_p(\operatorname{spt}\|\Delta\| \cap B^3(p,r)), \theta Y) < \varsigma$ *(resp., replace* θY *by* θT*) for some* $\theta \in \mathrm{SO}(3)$,

then there exists a 2*-varifold* Γ *with* $\Gamma = \lim_{k\to\infty} |\delta F_k'|$, $F_k' \in \Phi$ *such that*

(1) $\operatorname{spt}\|\Gamma\| \sim B^3(p,r) \subset \operatorname{spt}\|\Delta\| \sim B^3(p,r)$ *and*
(2) $r^{-2}H^2(\operatorname{spt}\|\Gamma\| \cap B^3(p,r))\exp(3\xi(r)/\alpha) - \pi\Theta^2(\Delta,p) \leq (1-k/2)E(r) + l\xi(r)$, *where*

$$E(r) = (2r)^{-1}H^1(\operatorname{spt}\|\Delta\| \cap \partial B^3(p,r))\exp(3\xi(r)/\alpha) - \pi\Theta^2(\Delta,p).$$

Proof. Multiplying conclusion (2) of Lemma 10 by $\exp(3\xi(r)/\alpha)$, we get

$$\begin{aligned}&H^2(\operatorname{spt}\|\Gamma\|\cap B^3(p,r))\exp(3\xi(r)/\alpha)\\&\quad\le(1-k/2)(r/2)H^1(\operatorname{spt}\|\Delta\|\cap\partial B^3(p,r))\exp(3\xi(r)/\alpha)\\&\qquad+(k/2)\pi r^2\theta^2(\Delta,p)\exp(3\xi(r)/\alpha).\end{aligned}$$

The above conclusion (2) is equivalent to

$$\begin{aligned}&H^2(\operatorname{spt}\|\Gamma\|\cap B^3(p,r))\exp(3\xi(r)/\alpha)\\&\quad\le(1-k/2)(r/2)H^1(\operatorname{spt}\|\Delta\|\cap\partial B^3(p,r))\exp(3\xi(r)/\alpha)\\&\qquad+(k/2)\pi r^2\Theta^2(\Delta,p)+lr^2\xi(r).\end{aligned}$$

Hence it suffices to show

$$(k/2)\pi r^2\Theta^2(\Delta,p)\exp(3\xi(r)/\alpha)\le(k/2)\pi r^2\Theta^2(\Delta,p)+lr^2\xi(r).$$

Since $e^x\le 1+Ax$ for $A=3^{10}$ and $0\le x\le 10$, we have

$$\exp(3\xi(r)/\alpha)\le 1+A(3\xi(r)/\alpha)\quad\text{for } 3/10\le\alpha\le 1/3 \text{ and } 0\le r\le 1.$$

Therefore

$$\begin{aligned}&(k/2)\pi r^2\Theta^2(\Delta,p)\exp(3\xi(r)/\alpha)\\&\quad\le(k/2)\pi r^2\Theta^2(\Delta,p)+(k/2)\pi r^2\Theta^2(\Delta,p)3A\xi(r)/\alpha\\&\quad\le(k/2)\pi r^2\Theta^2(\Delta,p)+lr^2\xi(r)\end{aligned}$$

for $l=32A$, and hence the required result is proved.

Theorem 4 ($C^{1,\alpha}$ *regularity of singular set*). (1) *At every point p in* $\operatorname{spt}\|\Delta\|$, Δ *has a unique tangent cone.*

(2) $\sigma_Y(\Delta)$ *is a one-dimensional $C^{1,\alpha}$ submanifold.*

(3) *In a neighborhood of $p\in\sigma_Y(\Delta)$ (respectively $\sigma_T(\Delta)$),* $\operatorname{spt}\|\Delta\|$ *is the union of three (respectively six) $C^{1,\alpha/2}$ manifolds with boundary.*

Proof. Chapter IV of [13] proves the above regularity for sets which are $(\mathbf{M},\xi,\delta)$ minimal under the Lipschitz map. Remember however that competing surfaces of $\operatorname{spt}\|\Delta\|$ are not the images of $\operatorname{spt}\|\Delta\|$ under Lipschitz maps but are the varifolds which are the limit of $|\delta F_k|$ for some sequence $\{F_k\}$ in Φ. We outline below the alterations to the proof in §IV of [13] that need to be made to accommodate this.

First notice that [13, II.1, II.2, II.4, II.6, and III.1] correspond to Lemma 4, Corollary 2, Lemma 9, Corollary 4, and Lemma 10′ respectively. [13, II.5] holds in our setting if we replace S_i by $\mu_i(\operatorname{spt}\|\Delta\|)$.

[13, IV.1]: Use $(\mathbf{M},\xi,\delta)$ minimality of $\operatorname{spt}\|\Delta\|$ as is described right before Lemma 10′, and then apply Lemma 10′.

[13, IV.4(2)]: Lipschitz maps f and g can be replaced by diffeomorphisms with the same property.

[13, IV.7(1)]: Lipschitz maps considered here can be replaced by a Lipschitz map ψ which preserves the topology of F_k, i.e., $M \sim \psi(\delta F_k) \approx B^\circ$, $\lim_{k\to\infty} |\delta F_k| = \Delta$.

Theorem 5. *If M is irreducible, then there exists a fundamental domain F in Φ with least boundary area among all elements of Φ, i.e., $\delta F = \operatorname{spt}\|\Delta\|$.*

Proof. In view of Theorem 1. (2) (ii)′, it is clear that, for any convex domain $U \subset M$ disjoint from the singular set of $\operatorname{spt}\|\Delta\|$, each component of $\operatorname{spt}\|\Delta\| \cap U$ is an area minimizing disk. Since $\operatorname{spt}\|\Delta\|$ is $C^{1,\alpha/2}$ up to its singular curve, there exists a $d > 0$ such that there is a C^1 retract ξ of $\Delta_d \equiv \{x \in M\colon \operatorname{dist}(x, \operatorname{spt}\|\Delta\|) < d\}$ onto $\operatorname{spt}\|\Delta\|$. Then we can apply Lemmas 6, 7, and 8 repeatedly at an appropriately chosen set of finite points of $\operatorname{spt}\|\Delta\|$ (using a finite open subcover of M) in such a way that for each k we obtain $F_k^5 \in \Phi$ with

$$\delta F_k^5 \subset \Delta_{1/k} \quad \text{and} \quad \lim_{k\to\infty} |\delta F_k^5| = \Delta. \tag{5.11}$$

Letting $k \to \infty$, it follows from (5.11) that

$$i_*(\pi_1(M \sim \operatorname{spt}\|\Delta\|)) = 0,$$

where i is the inclusion map into M. Hence $M \sim \operatorname{spt}\|\Delta\| = p(F)$, or $\operatorname{spt}\|\Delta\| = \delta F$ for some fundamental domain F in Φ_0. Let κ_k be a map from B onto M with $\kappa_k(\partial B) = \delta F_k^5$. Then, for k with $1/k < d$, $\xi\kappa_k$ maps ∂B into $\operatorname{spt}\|\Delta\|$. From (5.11) we know that $\xi\kappa_k$ is surjective. Now we claim F is in Φ. First suppose F is not connected and a component of F is not homeomorphic to B°. Then, since ∂B is mapped onto $\operatorname{spt}\|\Delta\|$, there exists a connected subset X of $\operatorname{spt}\|\Delta\|$ such that two disjoint subsets X_1 and X_2 of δF_k^5 are mapped onto X by ξ. Assume X is the largest such component. If X is a point or a curve, then the tangent cone of Δ at any point of X cannot be $|D|$, $|Y|$, or $|T|$. If X is a surface, then the tangent cone at any interior point of X is a plane with density 2 ($= 2|D|$). Since this contradicts Lemma 9, each component of $M \sim \operatorname{spt}\|\Delta\|$ must be homeomorphic to B°.

Suppose $M \sim \operatorname{spt}\|\Delta\| = \bigcup_{i=1}^n W_i$, $n \geq 2$, $W_i \approx B^\circ$, and $W_i \cap W_j = \varnothing$ if $i \neq j$. Then, using irreducibility of M, we paste W_i's to each other in such a way that we can obtain $W \approx B^\circ$ with $\bigcup_{i=1}^n W_i \subset W$ and $\partial W \subset \operatorname{spt}\|\Delta\|$. This is a contradiction since W gives rise to a fundamental domain $\hat{F} \in \Phi$ with $p(\hat{F}) = W$ and $H^2(\delta\hat{F}) < \mathbf{M}(\Delta)$. Hence $M \sim \operatorname{spt}\|\Delta\|$ must be homeomorphic to B°. Clearly F is not reducible. Therefore $F \in \Phi$.

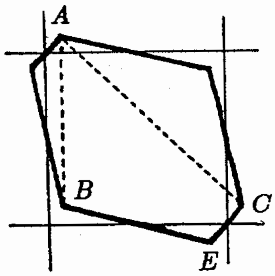


FIGURE 8

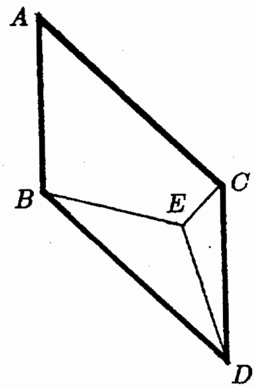


FIGURE 9

Appendix

1. Given a flat $\mathbf{T}^2$, not necessarily a square torus, we know from the proposition of §4 that a hexagon is a minimizing fundamental domain of the $\mathbf{T}^2$. Cut this hexagon along the dotted lines AB and AC (Figure 8).

After translating and pasting, we get another fundamental domain which is a parallelogram $ABCD$ (Figure 9). Note that the boundary length of the minimizing hexagon is twice the sum of the lengths of EB, EC, ED, and that E is a triple point joining the vertices of the acute triangle BCD. Since there exists one and only one triple point in an acute triangle we conclude that there exists a unique minimizing fundamental domain of flat $\mathbf{T}^2$ up to translations.

2. The standard projection map π from $S^2 \times S^1$ onto S^2 is an area-decreasing map: for any H^2-measurable set $X \subset S^2 \times S^1$,

$$H^2(X) \geq H^2(\pi(X)).$$

Moreover, for any fundamental domain F of $S^2 \times S^1$ in Φ_0, π maps $p(\partial F)$ onto S^2. Hence

$$H^2(p(\partial F)) \geq H^2(\pi(p(\partial F))) = H^2(S^2) = H^2(p(\partial(S^2 \times (0,1)))).$$

Thus $S^2 \times (0,1)$ is a minimizing fundamental domain of $S^2 \times S^1$.

3. 100 years ago, Sir William Thomson [14] considered a similar problem, periodic minimal partitioning of $\mathbf{R}^3$. His construction gives a candidate for the periodic division of $\mathbf{R}^3$ with minimum partitional area. But it has never been proved that his partitioning is the minimum. This candidate is a 14-faced domain whose boundary consists of six quadrilateral faces and eight hexagonal faces. This domain can be roughly obtained by truncating all vertices of regular octahedron. Here quadrilateral faces are flat, hexagonal faces are a monkey saddle, and each edge is a plane curve. The faces meet with the correct 120° angles along the edges and with the correct tetrahedral angles at the vertices. It turns out that Thomson's domain is a fundamental domain of a skew torus which is spanned by $4^{-1/3}(2,0,0)$, $4^{-1/3}(0,2,0)$, and $4^{-1/3}(1,1,1)$. We should mention that R. Kusner showed 14 is a lower bound for the number of faces of a minimizing fundamental domain of any flat torus. Notice that minimal partitioning of $\mathbf{R}^3$ is more general than periodic minimal partitioning of $\mathbf{R}^3$, and these two partitioning problems are more general than the problem of finding a minimizing fundamental domain of a specific flat torus.

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